Student Name: ______________________

Class Teacher: ______________________

St George Girls High School

Year 12

Common Test 3

June 2016

Mathematics Extension 1

General Instructions

- Working time – 70 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A BOSTES reference sheet is provided.

Section I	
Section II	
Q6	
Q7	
Q8	
Q9	
Q10	
Total Mark	

Total marks – 65

Section I: 5 marks
Attempt Questions 1 – 5
All questions are of equal value
Use the answer sheet provided

Section II: 60 marks
Attempt Questions 6 – 10.
All questions are of equal value.
In Questions 6 – 10, show relevant mathematical reasoning

Students are advised that this is a School Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

5 marks
Attempt Questions 1 – 5

Use the multiple choice answer sheet for Questions 1 – 5

1. If $f(x)=\frac{x+1}{x-2}$ then $f^{-1}(x)=$

(A) $\frac{1+2x}{x+1}$

(B) $\frac{1-2x}{x+1}$

(C) $\frac{1+2x}{x-1}$

(D) $\frac{1-2x}{x-1}$

2. The domain and range of $y=\sin^{-1}(2x+1)$ are:

(A) $D:\quad 0\le x\le 1$
$R:\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}$

(B) $D:\ -1\le x\le 0$
$R:\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}$

(C) $D:\quad 0\le x\le 1$
$R:\quad 0\le y\le\pi$

(D) $D:\ -1\le x\le 0$
$R:\quad 0\le y\le\pi$

3. Which of the following is a simplification of $\frac{1-\cos 2x}{\sin 2x}$:

(A) $1-\cot 2x$

(B) 1

(C) $\cot x$

(D) $\tan x$

Section I (cont'd)

4. $\displaystyle\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx =$

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

5. The expansion needed to show $\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$ is:

(A) $\sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

(B) $\sin(45° + 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

(C) $\sin(100° - 25°) = \sin 100° \cos 25° - \cos 100° \sin 25°$

(D) $\tan(45° + 30°) = \frac{\tan 45° \tan 30°}{1+\tan 45° \tan 30°}$

End of Section I

Section II

60 marks
Attempt Questions 6 – 10

Answer each question in a SEPARATE writing page.

Extra writing booklets are available.
In Questions 6 – 10, your responses should include relevant mathematical reasoning and/or calculations

Question 6 – (12 marks) – Use a SEPARATE writing page. Marks

a) Find:

(i) $\displaystyle\int 6\cos^2 4x\,dx.$ 2

(ii) $\displaystyle\int \frac{dx}{\sqrt{16-x^2}}.$ 1

b) Find $\frac{d}{dx}[\ln(\sec x + \tan x)]$. 2

c) (i) Show that $x^3 + x^2 - 6 = 0$ has a root between $x = 1$ and $x = 2$. 1

(ii) Use the method of halving the interval method twice, to find a better approximation of this root. 2

d) Sketch $y = \frac{1}{2}\cos^{-1} 3x$ stating its domain and range. 3

e) Find $\frac{dy}{dx}$ when $y = \tan^{-1} 4x$. 1

End of Question 6

Question 7 – (12 marks) – Use a SEPARATE writing page. **Marks**

a) Show $\sin(X+Y)+\sin(X-Y)=2\sin X\cos Y$ and hence, or otherwise, find **3**

$\int \sin 6x \cos 4x \, dx$.

b) Find the general solution, in radians, of $\sin 2x = \cos x$. **3**

c) The velocity of a particle is given by $\frac{dx}{dt}=e^{-0.6t}$, where $\frac{dx}{dt}$ is the velocity at time t.

(i) Explain why the particle never stops moving (that is velocity is never zero). **1**

(ii) If the particle starts at the origin write x as a function of t. **2**

(iii) For what value of t does $x=\frac{1}{2}$, answer to 1 decimal place. **2**

(iv) The limiting position of the particle occurs as $t \to \infty$. What is the limiting position of this particle? **1**

End of Question 7

Question 8 – (12 marks) – Use a SEPARATE writing page. **Marks**

a) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\theta)$ where θ is an acute angle and $R>0$. **2**

(ii) Hence, or otherwise, solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le \pi$. **2**

b) At any point on the curve $y=f(x)$ the gradient function is given by $\frac{dy}{dx}=\cos^2 x$. Find the value of $f\left(\frac{\pi}{3}\right)-f\left(\frac{\pi}{6}\right)$. **3**

c) (i) Find $\frac{d}{dx}\left(\cos^{-1}\left[\frac{x-4}{4}\right]\right)$. **2**

(ii) Hence, or otherwise, evaluate **2**

$$\int_2^4 \frac{1}{\sqrt{8x-x^2}}\,dx .$$

d) If $y=\sin^{-1}(e^x)$, find $\frac{dy}{dx}$. **1**

End of Question 8

Question 9 – (12 marks) – Use a SEPARATE writing page. **Marks**

a) Given $\cos\theta = \frac{1}{5}$ and $0 < \theta < \frac{\pi}{2}$, find the exact value of $\tan\frac{\theta}{2}$. 2

b) (i) Find the volume of the solid of revolution formed by rotating the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ about the x-axis. 2

(ii) Hence, find the volume of the solid formed by rotating $y = \sin^{-1}x$ between $y = 0$ and $y = \frac{\pi}{2}$ about the y-axis. 1

c) Find the equation of the tangent to the curve $y = \cot x$ at $x = \frac{\pi}{3}$. 3

d) The diagonal of a square is decreasing at a rate of $\frac{1}{4}$ m/s.

(i) Find the area (A) of a square with diagonal of length l. 1

(ii) Show that the rate of change of the area (A) is $\frac{dA}{dt} = -\frac{1}{4}l$ m^2/s. 1

(iii) Find the rate at which the area is decreasing when the area is 32 m^2. 1

(iv) What are the dimensions of the square when the area is decreasing at 4 m^2/s ? 1

End of Question 9

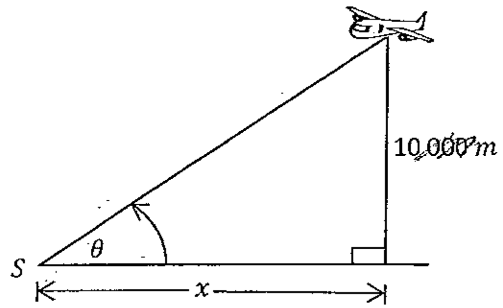
Question 10 – (12 marks) – Use a SEPARATE writing page. **Marks**

a) Given $y = \tan^{-1}\left[\frac{x+2}{1-2x}\right]$, show $\frac{dy}{dx} = \frac{1}{x^2+1}$. 3

b) Find $\int \frac{x}{\sqrt{16-x^2}}\,dx$ by using the substitution $x = 4\sin\theta$. 3

c) By letting $x = u - 3$, find $\int \frac{1}{x^2+6x+12}\,dx$. 3

d) A radar tracking station (S) is located at ground level below the path of an aircraft approaching it at 1080 km/hr, flying at a constant altitude of 10 000 metres. Show that the rate at which the radar beam to the aircraft is turning when the plane is directly above a point 3 km from S, is given by $\frac{3}{109}$ radians/second. 3



End of Examination

St George Girls

Year 12 common test 3. EXT. 1

June 2016 Sample solutions

Multiple choice

1. C
2. B
3. D
4. C
5. A

6. a) i) $\int 6\cos^2 4x \, dx$

$= 6\int \cos^2 4x \, dx$

$= 6\int \frac{1}{2}(1+\cos 8x)\,dx$

$= 3\int 1+\cos 8x \, dx$

$= 3\left[x + \frac{\sin 8x}{8}\right] + C$

$= \underline{\underline{3x + \frac{3}{8}\sin 8x + C}}$

ii) $\int \frac{1}{\sqrt{16-x^2}}\,dx$

$= \underline{\underline{\sin^{-1}\frac{x}{4} + C}}$

b.) $f(x) = \sec x + \tan x$

$f'(x) = \sec x \tan x + \sec^2 x$

$\frac{d}{dx}(\ln(\sec x + \tan x))$

$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}$

$= \underline{\underline{\sec x}}$

c. i) $x^3 + x^2 - 6 = 0$

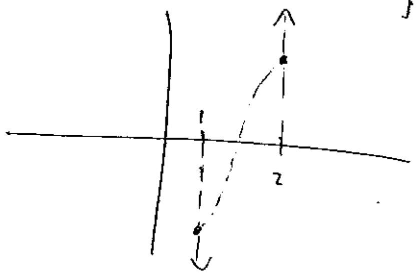
Sub in 1 for x

$1+1-6 = -4$ -ve.

Sub in 2 for x

$8+4-6 = 6$ +ve.

So that means, graphically



That means that, since the function is continuous for $x \in \mathbb{R}$, it must cross the x-axis somewhere between $x=1$ and $x=2$, thus our polynomial has a root between $x=1$ & $x=2$.

ii) Halving the interval.

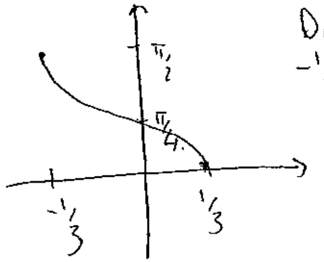
$x_1 = \frac{1+2}{2} = \frac{3}{2}$.

$P(\frac{3}{2}) = (\frac{3}{2})^3 + (\frac{3}{2})^2 - 6 = -\frac{3}{8} < 0$.

So therefore, a root lies between

$\underline{\underline{x = \frac{3}{2} \text{ and } 2.}}$

d) $y = \frac{1}{2}\cos^{-1} 3x$



Domain

$-\frac{1}{3} \le x \le \frac{1}{3}$

Range

$0 \le y \le \frac{\pi}{2}$.

e) $y = \tan^{-1} 4x$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)^2 + 1}$

$= \frac{4}{(4x)^2 + 1}$

$= \underline{\underline{\frac{4}{16x^2 + 1}}}$

$\sin(X+Y)+\sin(X-Y)=2\sin X\cos Y$ (RTP)

$\sin(X+Y)=\sin X\cos Y+\cos X\sin Y$ (1)

$\sin(X-Y)=\sin X\cos Y-\cos X\sin Y$ (2)

(1) + (2) $=\underline{\underline{2\sin X\cos Y}}$ (as required)

$\int \sin 6x\cos 4x$

$=\frac{1}{2}\int 2\sin 6x\cos 4x\,dx$

$=\frac{1}{2}\int \sin 10x+\sin 2x\,dx$

$=\frac{1}{2}\left(\frac{-\cos 10x}{10}-\frac{\cos 2x}{2}\right)+C$

$=\underline{\underline{-\frac{1}{2}\left(\frac{\cos 10x}{10}+\frac{\cos 2x}{2}\right)+C}}$

b) $\sin 2x=\cos x.$

$2\sin x\cos x=\cos x.$

$2\sin x=1$

$\sin x=\frac{1}{2}$

General Solution $\Rightarrow$ $\theta=n\pi+(-1)^n\sin^{-1}\left(\frac{1}{2}\right)$ $n\in\mathbb{Z}$

$\underline{\underline{\theta=n\pi+(-1)^n\left(\frac{\pi}{6}\right)}}$

c) $\frac{dx}{dt}=e^{-0.6t}.$

i) as $t\to\infty$, $\frac{dx}{dt}\to 0$

horizontal asymptote at 0

$\therefore$ particle never stops moving

ii) $\frac{dx}{dt}=e^{-0.6t}.$

$x=\int e^{-0.6t}\,dt.$

$=-\frac{5}{3}\int -\frac{3}{5}e^{-0.6t}$

$x=-\frac{5}{3}e^{-0.6t}+C$

at $t=0$, $x=0$.

$0=-\frac{5}{3}(1)+C$

$C=\frac{5}{3}$

$\underline{\underline{x=-\frac{5}{3}e^{-0.6t}+\frac{5}{3}}}$

iii) $\frac{1}{2}=-\frac{5}{3}e^{-0.6t}+\frac{5}{3}$

$-\frac{7}{6}=-\frac{5}{3}e^{-0.6t}$

$\frac{7}{10}=e^{-0.6t}$

$\ln\left(\frac{7}{10}\right)=-0.6t.$

$t=\frac{\ln\left(\frac{7}{10}\right)}{-0.6}$

$t=-\frac{5}{3}\ln\left(\frac{7}{10}\right)$

≈ 0.6

iv) as $t\to\infty$

$x=-\frac{5}{3}e^{-\infty}+\frac{5}{3}$

$x=\frac{5}{3}.$

8. a) $\sqrt{3}\cos x - \sin x$.

as $R\cos(x+\theta)$

$= R\cos x\cos\theta - R\sin x\sin\theta$.

$R\cos\theta = \sqrt{3}$, $R\sin\theta = 1$

$\cos\theta = \sqrt{3}/R$.

$R = \sqrt{1+3} = 2$.

when $R=2$.
$2\sin\theta = 1$
$\sin\theta = \pi/6$.
$\Rightarrow$ $\underline{2\cos(x+\pi/6)}$

ii) $\sqrt{3}\cos x - \sin x = 1$

$\Rightarrow 2\cos(x+\pi/6) = 1$.

$\cos(x+\pi/6) = \frac{1}{2}$

$0 \le x \le \pi$

$\pi/6 \le x + \pi/6 \le 7\pi/6$. } parameters

$x + \pi/6 = \pi/3$, $5\pi/3$

but $\pi/6 \le x+\pi/6 \le 7\pi/6$

$\therefore x + \pi/6 = \pi/3$

$\underline{x = \pi/6}$

b) $\frac{dy}{dx} = \cos^2 x$

$f(x) = \int \cos^2 x \, dx$

$= \int \frac{1}{2}(1+\cos 2x)dx$

$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

$\therefore f(\pi/3) = \frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) + C$

$= \pi/6 + \sqrt{3}/8$

$f(\pi/6) = \frac{1}{2}\left(\pi/6 + \frac{\sqrt{3}}{4}\right) + C$

$= \pi/12 + \sqrt{3}/8 + C$.

$f(\pi/3) - f(\pi/6)$

$= \pi/6 + \cancel{\sqrt{3}/8} + \cancel{C} - \pi/12 - \cancel{\sqrt{3}/8}$

$= \pi/6 - \pi/12 = \underline{\pi/12}$.

C i) $\frac{d}{dx}\left(\cos^{-1}\left[\frac{x-4}{4}\right]\right)$

$f(x) = \frac{x-4}{4} = \frac{1}{4}x - 1$

$f'(x) = \frac{1}{4}$

$\frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$

$= \frac{-1/4}{\sqrt{1-\left(\frac{x-4}{4}\right)^2}}$

$= \frac{-1}{\sqrt{8x-x^2}}$

ii) $\int_2^4 \frac{1}{\sqrt{8x-x^2}} dx$

$= -\int_2^4 \frac{-1}{\sqrt{8x-x^2}} dx$

$= -\cos^{-1}\left(\frac{x-4}{4}\right)\Big|_2^4$

$= -(\cos^{-1}0 - \cos^{-1} -\tfrac{1}{2})$

$= \underline{\pi/6}$.

d) $y = \sin^{-1} e^x$

$y' = \frac{e^x}{\sqrt{1-(e^x)^2}}$

$= \underline{\frac{e^x}{\sqrt{1-e^{2x}}}}$

a) $\cos\theta = \frac{1}{5}$

$0 < \theta < \pi/2$.

let $t = \tan\frac{\theta}{2}$.

$\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1}{5}$

$1 - t^2 = \frac{1}{5}t^2 + \frac{1}{5}$

$5 - 5t^2 = t^2 + 1$

$4 = 6t^2$

$t = \pm\sqrt{2/3}$

$\therefore \tan\frac{\theta}{2} = \pm\sqrt{2/3}$

b. i) $V = \pi \int_0^{\pi/2} (\sin x)^2 \, dx$.

$= \frac{\pi}{2}\int_0^{\pi/2} 1 - \cos 2x \, dx$

$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$= \pi^2/4 \, u^3$

ii) $y = \sin^{-1} x$

Inverse. $x = \sin^{-1} y$

$y = \sin x$ with parameters $y=0$ and $y = \frac{\pi}{2}$.

ie same parameters as i)

$\therefore$ volume $= \frac{\pi^2}{4} u^3$

c) $y = \cot x$ $\quad \frac{dy}{dx} = -\operatorname{cosec}^2 x$

at $x = 3$:

$\frac{dy}{dx} = -\frac{4}{3}$.

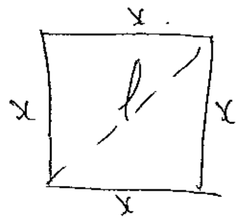
at $y = \cot(x)$

$y = \frac{1}{\sqrt{3}}$

$y - \frac{1}{\sqrt{3}} = -\frac{4}{3}\left(x - \frac{\pi}{3}\right)$

$12x + 9y - (4\pi + 3\sqrt{3}) = 0$

d)



i) $A = x^2$

$A = \left(\frac{l}{\sqrt{2}}\right)^2$

$= \frac{l^2}{2}$

ii) $\frac{dA}{dt} = \frac{dA}{dl} \times \frac{dl}{dt}$

$\frac{dA}{dl} = l$.

$\frac{dl}{dt} = -\frac{1}{4}$ (given).

$= -\frac{1}{4} l \ m^2/s$.

iii) at $A = 32$

$32 = \frac{l^2}{2}$

$l^2 = 64$

$l = 8$.

$= -\frac{1}{4}(8) \ m^2/s$

$= -2 \ m^2/s$.

iv) at $\frac{dA}{dt} = -4$

$-4 = -\frac{1}{4} l$

$l = 16$

$x = \frac{l}{\sqrt{2}}$

$x = \frac{16}{\sqrt{2}}$.

Dimensions $= \frac{16}{\sqrt{2}} \times \frac{16}{\sqrt{2}}$.

10. a) $y = \tan^{-1}\left[\frac{x+2}{1-2x}\right]$

$f(x) = \frac{x+2}{1-2x}$.

$u = x+2$

$u' = 1$

$v = 1-2x$

$v' = -2$.

$\therefore f'(x) = \frac{5}{(1-2x)^2}$.

$\frac{dy}{dx} = \frac{f'(x)}{(f(x))^2 + 1}$

$= \frac{1}{x^2+1}$

b) $\int \frac{x}{\sqrt{16-x^2}} \, dx$

let $x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$

$dx = 4\cos\theta \, d\theta$

$\int \frac{4\sin\theta}{\sqrt{16-(16\sin^2\theta)}} 4\cos\theta \, d\theta$

$= \int \frac{16\sin\theta\cos\theta \, d\theta}{\sqrt{16(1-\sin^2\theta)}}$

$= \int \frac{16\sin\theta\cos\theta \, d\theta}{4\cos\theta}$

$= \int 4\sin\theta \, d\theta$

$= -4\cos\theta + C$.

$= -\sqrt{16-x^2} + C$.

c) $\int \frac{1}{x^2+6x+12} \, dx$

let $x = u-3$. $\quad du = dx$

$\int \frac{1}{(u-3)^2 + 6(u-3) + 12}$

$= \int \frac{1}{u^2+3} \, du$.

$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{u}{\sqrt{3}} + C$

$= \Rightarrow \frac{1}{\sqrt{3}} \tan^{-1} \frac{x+3}{\sqrt{3}} + C$

d. $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$.

$\tan\theta = \frac{10}{x}$

$\theta = \tan^{-1} \frac{10}{x}$

$f(x) = 10x^{-1}$

$f'(x) = \frac{-10}{x^2}$.

$$\frac{d\theta}{dx} = \frac{\frac{-10}{x^2}}{\frac{100}{x^2} + 1}$$

$$= \frac{-10}{100 + x^2}$$

$\therefore \frac{d\theta}{dt} = \frac{-10}{100+x^2} \times \frac{3}{10}$ when $x = 3$.

$\frac{-10}{109} \times \frac{3}{10}$

$= \frac{-3}{109}$ rad / second.