YEAR 12 EXTENSION 1 TEST 31-8-05

Projectiles, Probability, Permutations and Combinations, Binomial Theorem.

Name______________________Class________

Instructions: Show all necessary working throughout the test on A4 paper.

Begin a new page as specified. **Time allowed: 50 minutes**

[Begin a new page] HRK

1.

(a) Find the value of the term independent of x in the expansion of $(2x - \frac{1}{x^2})^{12}$ 3

(b) By comparing coefficients of x^4 in both sides of

$(1 + x)^4(1 + x)^4 = (1 + x)^8$, show that

$\sum_{k=0}^{4} ({}^4C_k)^2 = {}^8C_4$ 3

(c) (i) Write down the Binomial expansion of $(1 + x)^n$ in ascending powers of x 1

Hence show that ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$ 2

(ii) Write down the expanded form of $\sum_{k=1}^{n-1} {}^nC_k$ 1

(iii) Show that ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} = 2^n - 2$ 1

(d) A machine is known to produce items of which 5% are too short and 95% are satisfactory. A random sample of twenty items is taken from the production of the machine.

Find the probability (correct to two decimal places) that:

(i) all of these items are satisfactory 1

(ii) at least eighteen of these items are satisfactory 3

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2. (a) The letters of the term "DELICIOUS FEAST" are arranged randomly in a row:

(i) prove that the number of different arrangements is 10 897 286 400 2

(ii) determine the number of ways that the vowels and consonants can alternate. 2

(b) At a round table there are 3 boys and 7 girls.

(i) In how many ways can the 10 people be seated at random? 1

(ii) If 3 of the girls wish to be seated next to one another, in how many ways can this seating arrangement be accommodated? 2

(iii) If a particular girl Anna is not to be seated between two particular boys Alexander and James, in how many ways can this seating arrangement be accommodated? 3

(c) Consider a pack of 40 playing cards consisting of the colours Red, Blue, Yellow and Green, with cards numbered from 1 to 10 for each colour. If five cards are dealt at random from the pack find:

(i) The total number of five card arrangements. 1

(ii) The probability of receiving three 4's and two 9's. 2

(iii) The probability of receiving five cards whose numbers are consecutive e.g. 3,4,5,6 and 7. 2

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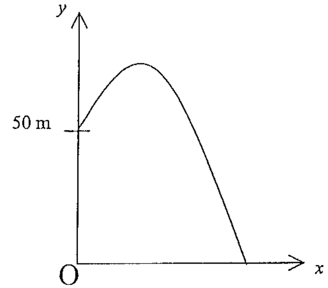
3. (a)



The diagram shows the path of a ball that is projected from the top of a tower 50 metres high. Its position t seconds after it is thrown is given by the equations:
$x = 20t$ and $y = 50 + 15t - 5t^2$ where the origin O is on the ground directly below the point of projection.

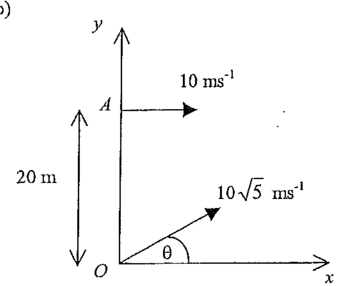
(i) Find the speed of projection **2**

(ii) Find the length of time before the ball strikes the ground. **1**

(iii) Calculate the maximum height above the ground reached by the ball. **2**

(iv) At what angle to the horizontal in the positive direction of the x-axis does the ball strike the ground? Give your answer to the nearest degree. **2**

(b)



OA is a vertical building of height 20 metres. A particle is projected horizontally from A with speed 10ms^{-1}. At the same instant another particle is projected from O with speed $10\sqrt{5}$ ms^{-1} at an angle θ above the horizontal. The two particles travel in the same plane of motion. Take $g = 10$ ms^{-1}.

(i) Derive expressions for the horizontal and vertical displacements relative to O of each particle after t seconds. **4**

(ii) Show that if the two particles collide, then they do so after 1 second. **2**

(iii) Show that if the two particles collide, when they do so their paths of motion are perpendicular to each other. **2**

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1(a) $T_r = {}^{12}C_{r-1}(2x)^{13-r}\left(-\frac{1}{x^2}\right)^{r-1}$

$= {}^{12}C_{r-1}\, 2^{13-r} x^{13-r} (-1)^{r-1} x^{-2r+2}$

$= {}^{12}C_{r-1}\, 2^{13-r} (-1)^{r-1} x^{15-3r}$

For term independent of x:

$15 - 3r = 0 \quad \therefore r = 5$

$\therefore T_5 = {}^{12}C_4\, 2^8 (-1)^4$

$= 126720$

$\Rightarrow$ term independent of x is 126720

(b) RHS $= (1+x)^8$

$= {}^8C_0 + {}^8C_1 x + \dots + {}^8C_4 x^4 + \dots + {}^8C_8 x^8$

LHS $= (1+x)^4(1+x)^4$

$= ({}^4C_0 + {}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4) \cdot ({}^4C_0 + {}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4)$

Comparing corr. coeffs of x^4 of LHS and RHS: ${}^8C_4 = {}^4C_0\,{}^4C_4 + {}^4C_1\,{}^4C_3 + {}^4C_2\,{}^4C_2 + {}^4C_3\,{}^4C_1 + {}^4C_4\,{}^4C_0$

$\therefore {}^8C_4 = ({}^4C_0)^2 + ({}^4C_1)^2 + ({}^4C_2)^2 + ({}^4C_3)^2 + ({}^4C_4)^2$

$= \sum_{k=0}^{4} ({}^4C_k)^2$

(c) (i) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

let $x = 1$ $\therefore (1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

$\therefore 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

(ii) $\sum_{k=1}^{n-1} {}^nC_k = {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$

(iii) From (i) and (ii)

$2^n = {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1} + {}^nC_0 + {}^nC_n$

Now as ${}^nC_0 = {}^nC_n = 1$

$\therefore 2^n - 2 = {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$

(d) Consider $(q+p)^{20}$

where $p = P(\text{satisfactory}) = 0.95$

$q = P(\overline{\text{satisfactory}}) = 0.05$

(i) $P(\text{all satisfactory}) = {}^{20}C_{20}\, p^{20}$

$= (0.95)^{20}$

$= 0.36$ (2dp)

(ii) $P(\text{at least 18 are satisfactory})$

$= P(18 \text{ sats}) + P(19 \text{ sats}) + P(20 \text{ sats})$

$= {}^{20}C_{18}\, q^2 p^{18} + {}^{20}C_{19}\, q\, p^{19} + {}^{20}C_{20}\, p^{20}$

$= {}^{20}C_{18}(0.05)^2(0.95)^{18} + {}^{20}C_{19}(0.05)(0.95)^{19} + {}^{20}C_{20}(0.95)^{20}$

$= 0.92$ (2dp)

2 (a) 'DELICIOUS FEAST' has 14 letters consisting of the vowels 'E' × 2, 'I' × 2, 'A', 'O' and 'U' and the consonants 'D', 'L', 'C', 'S' × 2, 'F' and 'T'.

(i) No. of different arrangements

$$= \frac{14!}{2!.2!.2!}$$

$$= 10\,897\,286\,400$$ ✓

(ii) If vowels and consonants alternate ∴ no. of arrangements

$$= \frac{2 \times 7! \times 7!}{2!.2!.2!}$$ ✓

$$= 6\,350\,400$$ ✓

(b) $\{3B, 7G\}$

(i) No. of ways $= 1 \times 9!$

$= 362\,880$ ✓

(ii) Consider the 3 girls as a unit.

This is achieved in $3!$ ways.

This leaves $(10-3)+1 = 8$ people units ✓

∴ No. of ways $= 1 \times 7! \times 3!$

$= 30240$ ✓

(iii) No of ways without restrictions $= 9!$

No. of ways Anna between Alexander & James ✓

$= 1 \times 2! \times 7!$

∴ No. of ways Anna not between Alex & James

$= 9! - 1 \times 2! \times 7!$

$= 352\,800$ ✓

(c) (i) No. of arrangements $= {}^{40}C_5$ ✓

$= 658\,008$

(ii) $P(3 \times 4\text{'s and } 2 \times 9\text{'s}) = \dfrac{{}^4C_3 \times {}^4C_2}{{}^{40}C_5}$ ✓

$= \dfrac{24}{658008}$ ✓

$= \dfrac{1}{27417}$ (or $\doteqdot 3.65 \times 10^{-5}$)

(iii) $P(\text{numbers are consecutive}) = \dfrac{(10-5+1) \times ({}^4C_1)^5}{{}^{40}C_5}$ ✓

$= \dfrac{6144}{658008}$

$= \dfrac{384}{41125}$ ✓

(OR $\doteqdot 9.34 \times 10^{-3}$)

3 (a) $x = 20t$ — (1)

$y = 50 + 15t - 5t^2$ — (2)

(i) $v^2 = \dot{x}^2 + \dot{y}^2$

$= (20)^2 + (15 - 10t)^2$

At speed of projection $t = 0$

$\therefore v^2 = 20^2 + 15^2$

$= 625$

$\therefore v = 25$

∴ Speed of projection is $25\ \text{ms}^{-1}$.

(ii) Ball strikes ground when $y = 0$

$\therefore 0 = 50 + 15t - 5t^2$

$\therefore 0 = 5t^2 - 15t - 50$

$0 = 5(t^2 - 3t - 10)$

$0 = 5(t-5)(t+2)$

$\therefore t = 5 \quad (t \geq 0)$

∴ Ball strikes ground after 5 secs.

(iii) At max. hgt $\dot{y} = 0$

$\therefore 15 - 10t = 0$

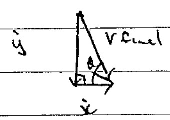
$\therefore t = 1\frac{1}{2}$

when $t = 1\frac{1}{2}$ $\quad y = 50 + 15(1\frac{1}{2}) - 5(1\frac{1}{2})^2$

$= 61\frac{1}{4}$

∴ Max. height reached by ball is $61\frac{1}{4}$ m.

(iv)



On ground: $\tan\theta = \left|\dfrac{\dot{y}}{\dot{x}}\right|$

$= \left|\dfrac{15 - 10t}{20}\right|$

At ground $t = 5$ $\quad \therefore \tan\theta = \left|\dfrac{-35}{20}\right|$

$\therefore \angle\theta \doteqdot 60°$ (to nearest deg)

∴ Ball strikes ground at approx. 120° to the horizontal in the positive direction of the x-axis.

(b)(i) For particle projected from A:

$\dot{x} = 10, \ \dot{y} = 0$

$\therefore \ddot{x} = 0, \ \ddot{y} = -10$

$\therefore \dot{x} = c_1, \ \dot{y} = -10t + c_2$

when $t = 0$ $\quad \dot{x} = 10, \ \dot{y} = 0$

$\therefore c_1 = 10, \ c_2 = 0$

$\therefore \dot{x} = 10, \ \dot{y} = -10t$

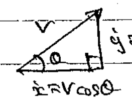
$\therefore x = 10t + c_3, \ y = -5t^2 + c_4$

when $t = 0$ $\quad x = 10 \quad y = 20$

$\therefore 10 = c_3, \ 20 = c_4$

$\therefore x = 10t + 10, \ y = -5t^2 + 20$

For particle projected from O:



$\therefore \ddot{x} = 0, \ \ddot{y} = -10$

$\therefore \dot{x} = c_1, \ \dot{y} = -10t + c_2$

when $t = 0$ $\quad \dot{x} = V\cos\theta \quad \dot{y} = V\sin\theta$

$\therefore V\cos\theta = c_1, \ V\sin\theta = c_2$

$\therefore \dot{x} = V\cos\theta, \ \dot{y} = -10t + V\sin\theta$

$\therefore x = Vt\cos\theta + c_3, \ y = -5t^2 + Vt\sin\theta + c_4$

when $t = 0$ $\quad x = 0, \ y = 0$

$\therefore c_3 = c_4 = 0$

$\therefore x = Vt\cos\theta, \ y = -5t^2 + Vt\sin\theta$

Now as $V = 10\sqrt{5}$

$\therefore x = 10\sqrt{5}\,t\cos\theta, \ y = -5t^2 + 10\sqrt{5}\,t\sin\theta$

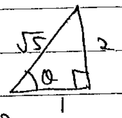
(ii) At point of collision of particles

$10t = 10\sqrt{5}t \cos\theta$ — (1)

and $-5t^2+20 = -5t^2+10\sqrt{5}t\sin\theta$ — (2)

Now from (1) $\cos\theta = \frac{1}{\sqrt{5}}$

$\therefore$ Triad is:



$\therefore \sin\theta = \frac{2}{\sqrt{5}}$ sub into (2)

$\therefore 20 = 10\sqrt{5}t \cdot \frac{2}{\sqrt{5}}$

$\therefore t = 1$

$\Rightarrow$ If particles collide they do so after 1 second.

(iii) Now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$\therefore$ For particle at A: $\frac{dy}{dx} = -10t \cdot \frac{1}{10}$

$= -t$

At $t=1$ $\frac{dy}{dx} = -1 = m_{\text{tangent}_A}$

For particle at O: $\frac{dy}{dx} = \frac{-10t+10\sqrt{5}\sin\theta}{10\sqrt{5}\cos\theta}$

At $t=1$ $\frac{dy}{dx} = \frac{-10+10\sqrt{5}\left(\frac{2}{\sqrt{5}}\right)}{10\sqrt{5}\left(\frac{1}{\sqrt{5}}\right)}$

$= \frac{10}{10}$

$= 1 = m_{\text{tangent}_O}$

As $m_{\text{tangent}_A} \cdot m_{\text{tangent}_O} = -1$

$\Rightarrow$ paths of motion are perpendicular to each other at time of collision.