NSW EDUCATION STANDARDS AUTHORITY

2017 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

1260

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which polynomial is a factor of $x^3 - 5x^2 + 11x - 10$?

A. $x - 2$

B. $x + 2$

C. $11x - 10$

D. $x^2 - 5x + 11$

2 It is given that $\log_a 8 = 1.893$, correct to 3 decimal places.

What is the value of $\log_a 4$, correct to 2 decimal places?

A. 0.95

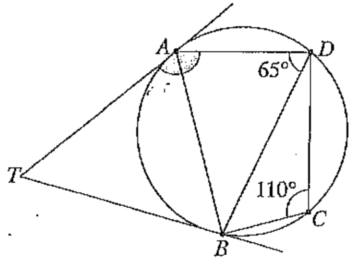
B. 1.26

C. 1.53

D. 2.84

3 The points A, B, C and D lie on a circle and the tangents at A and B meet at T, as shown in the diagram.

The angles BDA and BCD are $65°$ and $110°$ respectively.



What is the value of $\angle TAD$?

A. 130°

B. 135°

C. 155°

D. 175°

4 What is the value of $\tan\alpha$ when the expression $2\sin x - \cos x$ is written in the form $\sqrt{5}\sin(x-\alpha)$?

A. -2

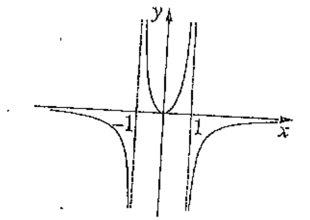
B. $-\frac{1}{2}$

C. $\frac{1}{2}$

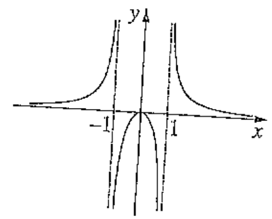
D. 2

5 Which graph best represents the function $y = \dfrac{2x^2}{1-x^2}$?

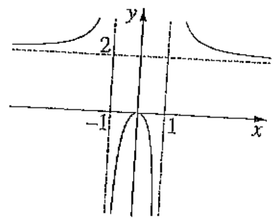
A.



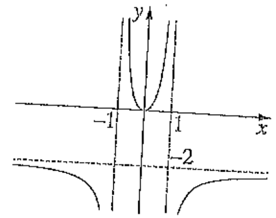
B.



C.



D.



6 The point $P\left(\dfrac{2}{p}, \dfrac{1}{p^2}\right)$, where $p \neq 0$, lies on the parabola $x^2 = 4y$.

What is the equation of the normal at P?

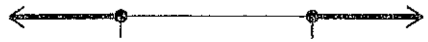
A. $py - x = -p$

B. $p^2y + px = -1$

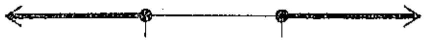
C. $p^2y - p^3x = 1 - 2p^2$

D. $p^2y + p^3x = 1 + 2p^2$

7 Which diagram represents the domain of the function $f(x)=\sin^{-1}\left(\frac{3}{x}\right)$?

A. 

B. 

C. 

D. 

8 A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of 5 cm s^{-1}.

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. 25π cm^2 s^{-1}

B. 30π cm^2 s^{-1}

C. 150π cm^2 s^{-1}

D. 225π cm^2 s^{-1}

9 When expanded, which expression has a non-zero constant term?

A. $\left(x+\frac{1}{x^2}\right)^7$

B. $\left(x^2+\frac{1}{x^3}\right)^7$

C. $\left(x^3+\frac{1}{x^4}\right)^7$

D. $\left(x^4+\frac{1}{x^5}\right)^7$

10 Three squares are chosen at random from the 3×3 grid below, and a cross is plac each chosen square.

What is the probability that all three crosses lie in the same row, column or diagonal?

A. $\frac{1}{28}$

B. $\frac{2}{21}$

C. $\frac{1}{3}$

D. $\frac{8}{9}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval from $A(-4, -4)$ to $B(1, 6)$ internally in the ratio $2:3$. **1**

Find the x-coordinate of P.

(b) Differentiate $\tan^{-1}(x^3)$. **2**

(c) Solve $\dfrac{2x}{x+1} > 1$. **3**

(d) Sketch the graph of the function $y = 2\cos^{-1}x$. **2**

(e) Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$.

(f) Find $\displaystyle\int \sin^2 x \cos x\,dx$.

Question 11 continues on page 8

(g) The probability that a particular type of seedling produces red flowers is $\frac{1}{5}$. Eight of these seedlings are planted.

(i) Write an expression for the probability that exactly three of the eight seedlings produce red flowers.

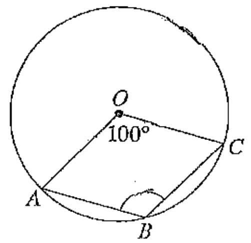
(ii) Write an expression for the probability that none of the eight seedlings produces red flowers.

(iii) Write an expression for the probability that at least one of the eight seedlings produces red flowers.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle AOC$ is $100°$. **2**



NOT TO SCALE

Find the size of $\angle ABC$, giving reasons.

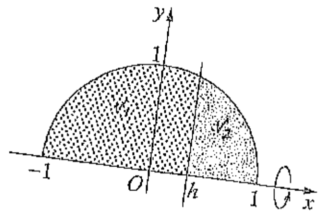
(b) (i) Carefully sketch the graphs of $y = |x+1|$ and $y = 3 - |x-2|$ on the same axes, showing all intercepts. **3**

(ii) Using the graphs from part (i), or otherwise, find the range of values of x for which **1**

$$|x+1| + |x-2| = 3.$$

Question 12 continues on page 10

(c) The region enclosed by the semicircle $y = \sqrt{1-x^2}$ and the x-axis is to b divided into two pieces by the line $x = h$, where $0 \le h < 1$.



The two pieces are rotated about the x-axis to form solids of revolution. The value of h is chosen so that the volumes of the solids are in the ratio $2:1$.

(i) Show that h satisfies the equation $3h^3 - 9h + 2 = 0$.

(ii) Given $h_1 = 0$ as the first approximation for h, use one application of Newton's method to find a second approximation for h.

(d) At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$.

Find the acceleration of the particle as a function of x.

(e) Evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the x-axis in simple harmonic motion centred at the origin. **3**

When $x = 2$ the velocity of the particle is 4.

When $x = 5$ the velocity of the particle is 3.

Find the period of the motion.

(b) Let n be a positive EVEN integer.

(i) Show that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$. **2**

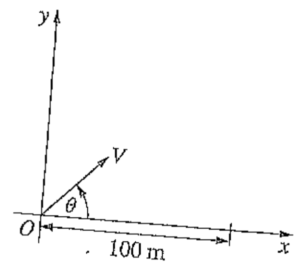
(ii) Hence show that **1**

$$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right].$$

(iii) Hence show that $\binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n} = n2^{n-3}$. **2**

Question 13 continues on page 12

(c) A golfer hits a golf ball with initial speed $V\ \mathrm{m\,s^{-1}}$ at an angle θ to the horizontal. The golf ball is hit from one side of a lake and must have a horizontal range of 100 m or more to avoid landing in the lake.



Neglecting the effects of air resistance, the equations describing the motion of the ball are

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time in seconds after the ball is hit and g is the acceleration due to gravity in $\mathrm{m\,s^{-2}}$. Do NOT prove these equations.

(i) Show that the horizontal range of the golf ball is $\dfrac{V^2\sin 2\theta}{g}$ metres. **2**

(ii) Show that if $V^2 < 100g$ then the horizontal range of the ball is less than 100 m. **1**

It is now given that $V^2 = 200g$ and that the horizontal range of the ball is 100 m or more.

(iii) Show that $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$.

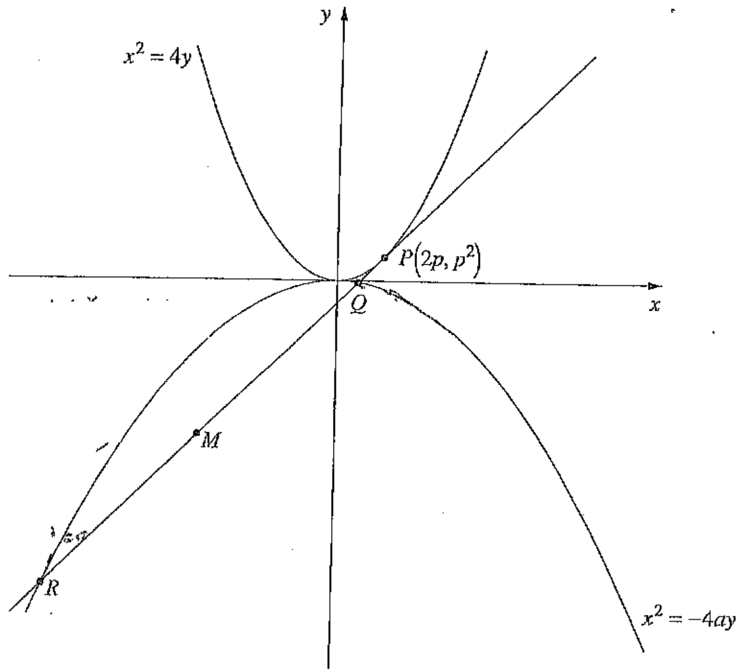
(iv) Find the greatest height the ball can achieve.

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that $8^{2n+1} + 6^{2n-1}$ is divisible by 7, for any integer $n \ge 1$. **3**

(b) Let $P\left(2p, p^2\right)$ be a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P meets the parabola $x^2 = -4ay$, $a > 0$, at Q and R. Let M be the midpoint of QR.



(i) Show that the x coordinates of R and Q satisfy **2**

$$x^2 + 4apx - 4ap^2 = 0.$$

(ii) Show that the coordinates of M are $\left(-2ap, -p^2(2a+1)\right)$. **2**

(iii) Find the value of a so that the point M always lies on the parabola $x^2 = -4y$. **2**

Question 14 continues on page 14

(c) The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

(i) By differentiating the product $F(t)e^{0.4t}$ show that **2**

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. **2**

(iii) The concentration of the drug increases to a maximum. **2**

For what value of t does this maximum occur?

Full Solutions

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$8 - 20 + 22 - 10$

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What is the value of $\log_a 4$, correct to 2 decimal places?

A. 0.95

B. 1.26

C. 1.53

D. 2.84

$3 \log 2 = 1.893$

$2 \log 2 = \frac{1.893}{3} \times 2$

$=$

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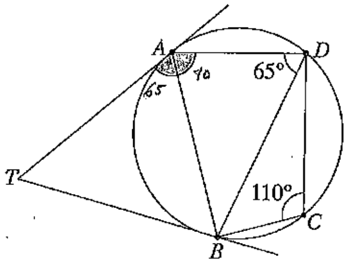

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C. 155°

D. 175°

4 What is the value of $\tan\alpha$ when the expression $2\sin x - \cos x$ is written in the form $\sqrt{5}\sin(x-\alpha)$?

$\tan\alpha = \frac{1}{2}$

$\alpha = \tan^{-1}(\frac{1}{2})$

A. -2

B. $-\frac{1}{2}$

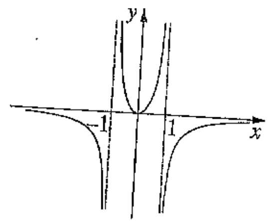
C. $\frac{1}{2}$

D. 2

5 Which graph best represents the function $y = \dfrac{2x^2}{1-x^2}$?

$= \quad -x^2+1 \,\overline{)\,2x^2}$ $\quad -2$ $\quad 2x^2$

A.



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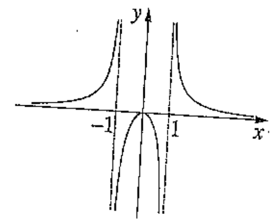
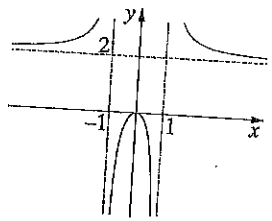
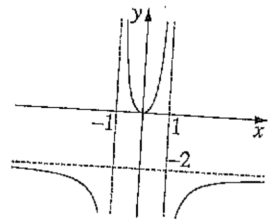


C.



D.



6 The point $P\left(\dfrac{2}{p}, \dfrac{1}{p^2}\right)$, where $p \neq 0$, lies on the parabola $x^2 = 4y$.

What is the equation of the normal at P?

$y' = \frac{2x}{4} = \frac{\frac{2}{p}}{2} = \frac{1}{p}$

$y - \frac{1}{p^2} = -p(x - \frac{2}{p})$

$p^2y - 1 = -p^3(x - \frac{2}{p})$

$= -p^3x + 2p^2$

A. $py - x = -p$

B. $p^2y + px = -1$

C. $p^2y - p^3x = 1 - 2p^2$

D. $p^2y + p^3x = 1 + 2p^2$

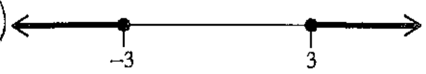
7 Which diagram represents the domain of the function $f(x)=\sin^{-1}\left(\frac{3}{x}\right)$? $-\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}$

$\sin \frac{3}{4}$

$\sin^{-1}(-1) = -\frac{\pi}{2}$

$\left(-\frac{3}{x}\right)$

A.



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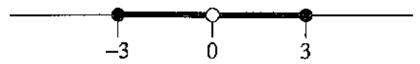

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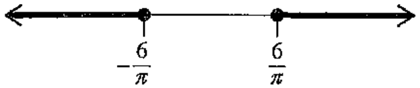
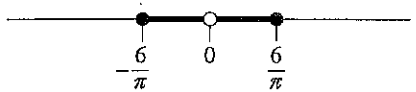


D.



8 A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of 5 cm s^{-1}.

$A = \pi r^2$

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. 25π cm^2 s^{-1}

B. 30π cm^2 s^{-1}

C. 150π cm^2 s^{-1}

D. 225π cm^2 s^{-1}

$\frac{dr}{dt} = 5$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

$= 2\pi r \cdot 5$

$= 10\pi (15)$

$= 150\pi$

9 When expanded, which expression has a non-zero constant term?

A. $\left(x+\frac{1}{x^2}\right)^7$

B. $\left(x^2+\frac{1}{x^3}\right)^7$

C. $\left(x^3+\frac{1}{x^4}\right)^7$

D. $\left(x^4+\frac{1}{x^5}\right)^7$

$T_{r+1} = {}^7C_r\, a^{7-r}\, b^r$

$(x^4)^{7-r}(x^{-5})^r = 0$

$28 - 4r - 5r = 0$

$(x^3)^{7-r}(x^{-4})^r$

$21 - 3r - 4r = 0$

$7r = 21$

10 Three squares are chosen at random from the 3 × 3 grid below, and a cross is place each chosen square.

What is the probability that all three crosses lie in the same row, column or diagonal?

A. $\frac{1}{28}$

B. $\frac{2}{21}$

C. $\frac{1}{3}$

D. $\frac{8}{9}$

$\frac{8}{{}^9C_3} = \frac{8 \times 3 \times 2}{9 \times 8 \times 7}$

$= \frac{2}{21}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

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Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval from $A(-4,-4)$ to $B(1,6)$ internally in the ratio $2:3$. **1**

Find the x-coordinate of P.

$\underline{x = -2}$

$-4 \quad 1$
$2 \quad 3$

$x = \frac{-12+2}{5} = -\frac{10}{5} = -2$

(b) Differentiate $\tan^{-1}(x^3)$. **2**

$y' = 3x^2 \cdot \frac{1}{1+x^6} = \frac{3x^2}{1+x^6}$

(c) Solve $\frac{2x}{x+1} > 1$. **3**

$\Rightarrow \frac{2x - x - 1}{x+1} > 0$

$\frac{x-1}{x+1} > 0$

$-1 \quad 1$

$\boxed{x > 1 \text{ or } x < -1}$

(d) Sketch the graph of the function $y = 2\cos^{-1}x$. **2**

(e) Evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$. **3**

$x = 3,\ u = 2$ $\quad \frac{dx}{du} = 2u$
$x = 0,\ u = 1$

$\int_1^2 \frac{u^2-1}{u} \cdot 2u\,du$

$= 2\left[\frac{u^3}{3} - u\right]_1^2 = 2\left[\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right]$

$= 2\left[\frac{7}{3} - 1\right]$

$= 2 \cdot \frac{4}{3} = \frac{8}{3}$

(f) Find $\int \sin^2 x \cos x\,dx$.

$u = \sin x$
$\frac{du}{dx} = \cos x$

$= \int u^2\,du$

$= \frac{u^3}{3} + c$

$= \frac{\sin^3 x}{3} + C$

Question 11 continues on page 8

(g) The probability that a particular type of seedling produces red flowers is $\frac{1}{5}$. Eight of these seedlings are planted.

(i) Write an expression for the probability that exactly three of the eight seedlings produce red flowers.

${}^8C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^5$

(ii) Write an expression for the probability that none of the eight seedlings produces red flowers.

$\left(\frac{4}{5}\right)^8$

(iii) Write an expression for the probability that at least one of the eight seedlings produces red flowers.

$1 - \left(\frac{4}{5}\right)^8$

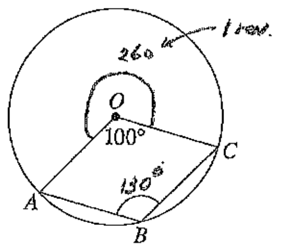
End of Question 11

$p = \frac{1}{5} \quad q = \frac{4}{5}$

$(p+q)^8$

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle AOC$ is $100°$. **2**



NOT TO SCALE

Find the size of $\angle ABC$, giving reasons. $= 130°$ ($\angle$ at centre $= 2\angle$ on the circumf)

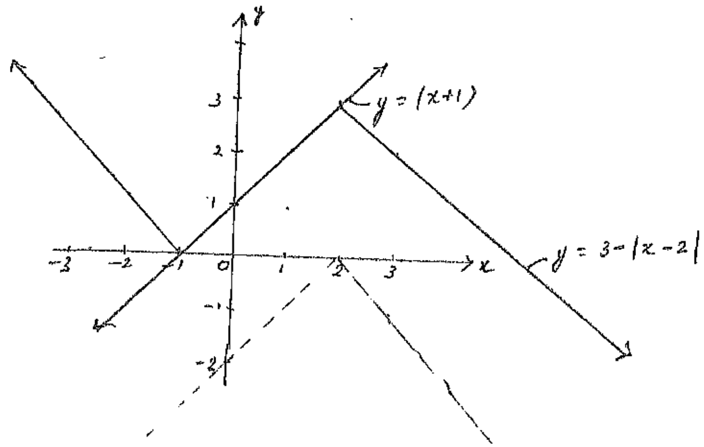
(b) (i) Carefully sketch the graphs of $y=|x+1|$ and $y=3-|x-2|$ on the same axes, showing all intercepts. **3**

(ii) Using the graphs from part (i), or otherwise, find the range of values of x for which **1**

$$|x+1|+|x-2|=3. \quad \Rightarrow \quad |x+1| = 3-|x-2|$$

$$\boxed{-1 \le x \le 2}$$

Question 12 continues on page 10



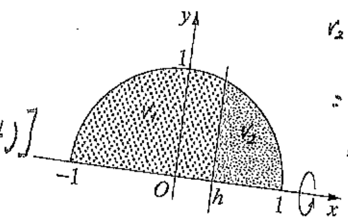
(c) The region enclosed by the semicircle $y=\sqrt{1-x^2}$ and the x-axis is to be divided into two pieces by the line $x=h$, where $0 \le h < 1$.

$V_1 = \pi\int_{-1}^{h} 1-x^2\,dx$

$= \pi\left[x-\frac{x^3}{3}\right]_{-1}^{h}$

$= \pi\left[\left(h-\frac{h^3}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$

$= \pi\left[h+\frac{2}{3}-\frac{h^3}{3}\right]$



$V_2 = \pi\int_{h}^{1} 1-x^2\,dx$

$= \pi\left[\left(1-\frac{1}{3}\right)-\left(h-\frac{h^3}{3}\right)\right]$

$= \pi\left[\frac{2}{3}-h+\frac{h^3}{3}\right]$

The two pieces are rotated about the x-axis to form solids of revolution. The value of h is chosen so that the volumes of the solids are in the ratio $2:1$.

$\frac{V_1}{V_2} = \frac{2}{1}$

$\frac{3h+2-h^3}{2-3h+h^3} = \frac{2}{1}$

$3h+2-h^3 = 4-6h+2h^3$

$3h^3-9h+2=0$ as req'd.

(i) Show that h satisfies the equation $3h^3-9h+2=0$.

(ii) Given $h_1=0$ as the first approximation for h, use one application of Newton's method to find a second approximation for h.

$f(h) = 3h^3-9h+2 \Rightarrow f(0)=2$

$f'(h) = 9h^2-9 \Rightarrow f'(0)=-9$

$\therefore h_2 = h_1 - \frac{f(h_1)}{f'(h_1)}$

$= 0-\frac{2}{-9}$

$= \frac{2}{9}$

(d) At time t the displacement, x, of a particle satisfies $t=4-e^{-2x}$. **3**

Find the acceleration of the particle as a function of x.

$e^{-2x} = 4-t$

$e^{2x} = \frac{1}{4-t}$

$2x = \ln\left(\frac{1}{4-t}\right)$

$x = \frac{1}{2}\ln\frac{1}{4-t}$

$\dot{x} = \frac{1}{2}\cdot\frac{-1}{4-t}$

$= \frac{-1}{8-2t} = -(8-2t)^{-1}$

$\ddot{x} = 1(8-2t)^{-2}\cdot(-2)$

$= \frac{-2}{(8-2t)^2}$

$= \frac{-2}{(8-2(4-e^{-2x}))^2}$

$= \frac{-2}{(2e^{-2x})^2}$

$= \frac{-2}{4e^{-4x}}$

$= \frac{-1}{2e^{-4x}}$

(e) Evaluate $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$. **2**

$\cos 2x = 1-2\sin^2 x$

$\Downarrow$

$\lim_{x\to 0}\frac{2\sin^2 x}{x^2}$

$= 2$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the x-axis in simple harmonic motion centred at the origin. **3**

When $x = 2$ the velocity of the particle is 4.

When $x = 5$ the velocity of the particle is 3.

Find the period of the motion.

$v^2 = n^2(a^2 - x^2)$

$4^2 = n^2(a^2 - 4)$ ... ①

$3^2 = n^2(a^2 - 25)$ ②

$\frac{16}{9} = \frac{a^2-4}{a^2-25}$

$16(a^2-25) = 9(a^2-4)$

$7a^2 = -36 + 400 = 364$

$a^2 = 52$

$a = \sqrt{52}$

$\therefore\ 9 = n^2(52-25)$

$n^2 = \frac{1}{3}$ $\quad n = \frac{1}{\sqrt{3}}$

$\therefore T = 2\sqrt{3}\pi$

(b) Let n be a positive EVEN integer.

(i) Show that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$. **2**

(ii) Hence show that **1**

$$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right].$$

(iii) Hence show that $\binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n} = n2^{n-3}$. **2**

Question 13 continues on page 12

(i) $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 \ldots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

$+\ (1-x)^n = {}^nC_0(-x)^0 + {}^nC_1(-x)^1 + {}^nC_2(-x)^2 \ldots + {}^nC_{n-1}(-x)^{n-1} + {}^nC_n(-x)^n$

$\therefore = 2\,{}^nC_0 + 2\,{}^nC_2 x^2 + \ldots + 2\,{}^nC_n x^n$ when n is even

$= 2\left[{}^nC_0 + {}^nC_2 x^2 + \ldots + {}^nC_n x^n\right]$

(ii) Diff (i) w.r.t. x

$n(1+x)^{n-1} + n(1-x)^{n-1} \cdot -1 = 4\,{}^nC_2 x + 8\,{}^nC_4 x^3 + \ldots + 2n\,{}^nC_n x^{n-1}$

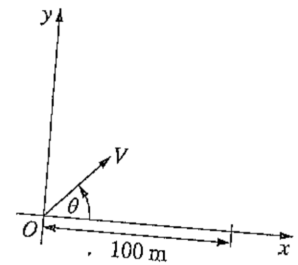
$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\,{}^nC_2 x + 4\,{}^nC_4 x^3 + \ldots + n\,{}^nC_n x^{n-1}\right]$

(iii) Let $x = 1$

$n \cdot 2^{n-1} = 2 \cdot 2\left[{}^nC_2 + 2\,{}^nC_4 + \ldots + \frac{n}{2}\,{}^nC_n\right]$

$n \cdot \frac{2^{n-1}}{2^2} = n \cdot 2^{n-3}$

(c) A golfer hits a golf ball with initial speed $V\ \mathrm{m\,s^{-1}}$ at an angle θ to the horizontal. The golf ball is hit from one side of a lake and must have a horizontal range of 100 m or more to avoid landing in the lake.



Neglecting the effects of air resistance, the equations describing the motion of the ball are

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time in seconds after the ball is hit and g is the acceleration due to gravity in $\mathrm{m\,s^{-2}}$. Do NOT prove these equations.

(i) Show that the horizontal range of the golf ball is $\frac{V^2\sin 2\theta}{g}$ metres. **2**

(ii) Show that if $V^2 < 100g$ then the horizontal range of the ball is less than 100 m. **1**

It is now given that $V^2 = 200g$ and that the horizontal range of the ball is 100 m or more.

(iii) Show that $\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$. **2**

(iv) Find the greatest height the ball can achieve. **2**

For greatest height

$\dot{y} = 0 = V\sin\theta - gt$

$t = \frac{V\sin\theta}{g}$

$\therefore\ y_{max} = \frac{V\sin\theta \cdot V\sin\theta}{g} - \frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2$

$= \frac{V^2\sin^2\theta}{2g}$

$\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$ $\quad \theta = \frac{\pi}{4} = \frac{3\pi}{12}$ $\quad y_{max} = \frac{V^2}{2g} \cdot \frac{1}{2}$ $\quad \boxed{= \frac{V^2}{4g}}$

End of Question 13

(i) $y = 0 = t\left(V\sin\theta - \frac{1}{2}gt\right)$ $\quad \therefore t = 0$ or $t = \frac{2V\sin\theta}{g}$

$\therefore\ x = R = V\cos\theta\left(\frac{2V\sin\theta}{g}\right) = \frac{V^2\sin 2\theta}{g}$ m. since $2\sin\theta\cos\theta = \sin 2\theta$

(ii) $V^2 < 100g$ $\quad \frac{V^2\sin 2\theta}{g} < \frac{100 g\sin 2\theta}{g} < 100$ since $-1 \le \sin 2\theta \le 1$

(iii) $V^2 = 200g$ $\quad R \ge 100$

$\therefore\ 200\sin 2\theta \ge 100$ $\quad 0 \le \theta \le \frac{\pi}{2}$

$\sin 2\theta \ge \frac{1}{2}$ $\quad 0 \le 2\theta \le \pi$

$\frac{\pi}{6} \le 2\theta \le \frac{5\pi}{6}$ $\quad \boxed{\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}}$

Question 14 (15 marks) Use a SEPARATE writing booklet.

$8^{2k+1} + 6^{2k-1} = 7M$

$8^{2k+3} + 6^{2k+1} = 7N$

(a) Prove by mathematical induction that $8^{2n+1} + 6^{2n-1}$ is divisible by 7, for any integer $n \geq 1$. **3**

LHS $= 8^{2k+1} \cdot 8^2 + 6^{2k-1} \cdot 6^2$

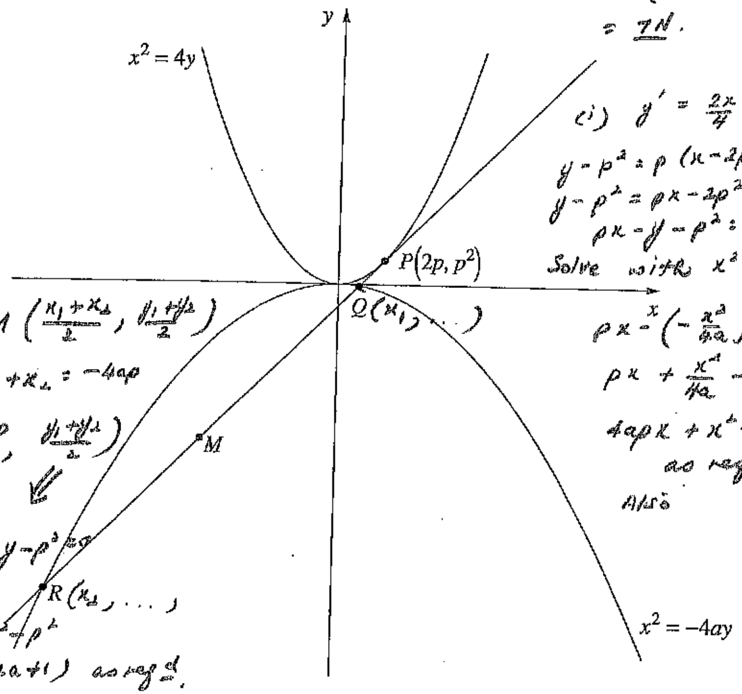
$= 64(7M - 6^{2k-1}) + 36 \cdot 6^{2k-1} = 64 \cdot 7M - 64m + 36m = -28m$

$= 7(64M - 4m)$

$= \underline{7N}$.

(b) Let $P(2p, p^2)$ be a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P meets the parabola $x^2 = -4ay$, $a > 0$, at Q and R. Let M be the midpoint of QR.



(i) $y' = \frac{2x}{4} = \frac{4p}{4} = p$

$y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$px - y - p^2 = 0 \quad \ldots ①$

Solve with $x^2 = -4ay$

$px - \left(-\frac{x^2}{4a}\right) - p^2 = 0$

$px + \frac{x^2}{4a} - p^2 = 0$

$4apx + x^2 - 4ap^2 = 0$ as req^d.

Also

(ii) $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

but $x_1 + x_2 = -4ap$

$\therefore \left(-\frac{4ap}{2}, \frac{y_1+y_2}{2}\right)$

sub into the line

$p\left(-\frac{4ap}{2}\right) - y - p^2 = 0$

$\therefore y = -2ap^2 - p^2$

$= -p^2(2a+1)$ as req^d.

(i) Show that the x coordinates of R and Q satisfy **2**

$$x^2 + 4apx - 4ap^2 = 0.$$

(ii) Show that the coordinates of M are $(-2ap, -p^2(2a+1))$. **2**

(iii) Find the value of a so that the point M always lies on the parabola $x^2 = -4y$. **2**

(iii) $x = -2ap$, $y = -p^2(2a+1)$ $\Rightarrow$ $4a^2p^2 = 4p^2(2a+1)$

$4a^2 = 8a + 4$

$a^2 - 2a - 1 = 0$

$a = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$

Question 14 continues on page 14

$50e^{-1} - 16.34$

$18 - 16.35$

$11.16 - 10.61$

Question 14 (continued)

(c) The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

$t = 2$	$t = 2.23$	$t = 3$
$F'(t) = 2.14$		$F'(t) = -4.45$

$\therefore$ Max

(i) By differentiating the product $F(t)e^{0.4t}$ show that **2**

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. **2**

(iii) The concentration of the drug increases to a maximum. **2**

For what value of t does this maximum occur?

(iii) $F'(t) = 50e^{-0.5t} - 0.4 \times 500(e^{-0.4t} - e^{-0.5t}) = 0$

$= 50e^{-0.5t} - 200e^{-0.4t} + 200e^{-0.5t} = 0$

$250e^{-0.5t} = 200e^{-0.4t}$

$e^{-0.1t} = \frac{4}{5}$

$e^{0.1t} = \frac{5}{4} \Rightarrow 0.1t = \ln\left(\frac{5}{4}\right) \Rightarrow \boxed{t = 10\ln\left(\frac{5}{4}\right) \approx 2.23 \text{ hrs}} \longrightarrow$ Test for max

End of paper

(i) $\frac{d}{dt}\left(F(t) \cdot e^{0.4t}\right) = F'(t) \cdot e^{0.4t} + 0.4F(t) \cdot e^{0.4t}$

$= \left[50e^{-0.5t} - 0.4F(t)\right]e^{0.4t} + 0.4F(t)e^{0.4t}$

$= 50e^{-0.1t} - 0.4F(t) \cdot e^{0.4t} + 0.4F(t)e^{0.4t}$

$= 50e^{-0.1t}$ as req^d.

(ii) $\int \frac{d}{dt}\left(F(t)e^{0.4t}\right)dt = \int 50e^{-0.1t}$

$F(t)e^{0.4t} = \frac{50e^{-0.1t}}{-0.1} + C$

At $t = 0$, $F(t) = 0$

$C = \frac{50}{0.1} = 500$

$\therefore F(t)e^{0.4t} = 500 - 500e^{-0.1t}$

$\therefore F(t) = 500\left(\frac{1 - e^{-0.1t}}{e^{0.4t}}\right)$

$= 500\left(e^{-0.4t} - e^{-0.5t}\right)$