Student N___

St Catherine's School
Waverley

Year 12 Mathematics Extension 1

Task 1

November 2016

Time allowed: 55 minutes plus 3 minutes reading time

Total marks: 41 marks

Weighting: 10%

There are 6 questions including 3 multiple choice questions.

INSTRUCTIONS

- Use 3 booklets. You may request more booklets if needed.
  - (i) Q1 – Q4
  - (ii) Q5
  - (iii) Q6
- Show all necessary working for questions 4, 5 and 6.
- Marks may be deducted for careless or badly arranged work.
- A formula sheet is provided on the back of this paper.
- Write using blue or black pen. Black pen is preferred.

Multiple choice	/ 3 marks
Question 4	/13 marks
Question 5	/13 marks
Question 6	/12 marks
Total	/41 marks

Section 1

3 marks

Attempt questions 1 – 3

1. If $t = \tan\frac{\theta}{2}$, what is the expression for $2\sin\theta - 2$?

   (A) $\frac{-2t^2+4t-2}{1+t^2}$　　(B) $\frac{4t-2+2t^2}{1+t^2}$

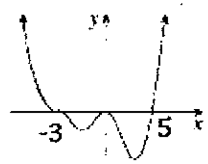
   (C) $\frac{-2+4t+t^2}{1+t^2}$　　(D) $\frac{-2t^2+4t-2}{1-t^2}$

2. When $P(x)$ is divided by $(x^2 + x - 2)$ the reminder is $(2x + 5)$. What is the reminder when $P(x)$ is divided by $(x - 1)$?

   (A) 0　　(B) 1

   (C) $(x + 2)$　　(D) 7

3. What is a possible equation of the graph below?



   (A) $y = 2(x-3)^2x^3(x+5)$　　(B) $y = -3(x+3)^3x^2(x-5)$

   (C) $y = 2(x+3)^3x^2(x-5)$　　(D) $y = -3(x-3)^2x^3(x+5)$

END OF SECTION 1

Section II

40 Marks

Attempt questions 4 – 6

Answer each question in a **SEPARATE writing booklet.** Extra writing booklets are available.

In Questions 4 – 6, your responses should include relevant mathematical reasoning and/or calculations.

Question 4 (13 marks) **Use a SEPARATE writing booklet.** **Marks**

(a) Prove the identity: **3**

$$\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$$

(b) Find the general solution of **3**

$$4sin^2x = 7cosx + 2$$

(c) $P(x) = x^3 + ax^2 + bx - 18$. Find the values of a and b if $(x+2)$ is a factor of $P(x)$ and when $P(x)$ is divided by $(x-1)$ the reminder is -24. **3**

(d) (i) Write $4\cos x + 3\sin x$ in the form $R\sin(x+\alpha)$ **2**

(ii) Hence, solve the equation $4\cos x + 3\sin x = -1$, for $0° \le x \le 360°$ **2**

END OF QUESTION 4

Question 5 (13 marks) **Use a SEPARATE writing booklet.** **Marks**

(a) $P(x) = x^3 - 2x^2 - 5x + 6$

(i) Show that $(x-1)$ is a factor of $P(x)$. **1**

(ii) Fully factorise $P(x)$. **2**

(iii) Sketch the polynomial $P(x)$ showing all intercepts. **1**

(iv) Hence or otherwise solve: **1**

$$\frac{(x-3)(x+2)}{(x-1)} \ge 0$$

(b) Prove by mathematical induction that **4**

$$\frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1} \text{ for all integers } n \ge 1.$$

(c) Solve the equation $4x^3 + 32x^2 + 79x + 60 = 0$ given that one root is equal to the sum of other two roots. **4**

END OF QUESTION 5

Question 6 (12 marks) **Use a SEPARATE writing booklet.** **Marks**

(a) **(i)** If $\tan\alpha = \frac{1}{3}$, show that $\tan 2\alpha = \frac{3}{4}$. **2**

(ii) If $\tan\beta = \frac{1}{7}$, show that $(2\alpha + \beta) = 45°$. **2**

(b) If $t = \tan\frac{\theta}{2}$, show that: **3**

$$\frac{\cot\theta}{1 - \operatorname{cosec}\theta} = \frac{t+1}{t-1}$$

(c) **Prove by mathematical induction that $7^n + 6^n$ is divisible by 13 for all odd positive integers n.** **3**

(d) Show that $(x-1)(x-2)$ is a factor of the polynomial **2**

$$P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + 2^n - 2^m.$$

END OF TASK

Yr 12 Maths Ext 1 task 1

Qn	Solutions	Marks	Comments: Criteria

1. $2\sin\theta - 2 = \frac{4t}{1+t^2} - 2$

$= \frac{4t - 2 - 2t^2}{1+t^2}$ (A)

2. $P(x) = Q(x)(x^2 + x - 2) + 2x + 5$

$= Q(x)(x+2)(x-1) + 2x + 5$

$P(1) = 2 + 5$

$= 7$ (D)

3. $y = 2(x+3)^3 x^2 (x-5)$ (C)

4. a) LHS: $\frac{1-\cos\alpha}{\sin\alpha} = \frac{1 - \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$

$= \frac{\cancel{2}\sin^{\cancel{2}}\frac{\alpha}{2}}{\cancel{2}\cancel{\sin\frac{\alpha}{2}}\cos\frac{\alpha}{2}}$

$= \tan\frac{\alpha}{2}$

$=$ RHS

b) $4\sin^2 x = 7\cos x + 2$

$4(1 - \cos^2 x) - 7\cos x - 2 = 0$

$4 - 4\cos^2 x - 7\cos x - 2 = 0$

$-4\cos^2 x - 7\cos x + 2 = 0$

$4\cos^2 x + 7\cos x - 2 = 0$

$4t^2 + 7t - 2 = 0$

$t = \frac{-7 \pm \sqrt{49+32}}{8} = \frac{-7 \pm 9}{8}$ $\frac{2}{8} = \frac{1}{4}$; $-\frac{16}{8} = -2$ ✗

$\cos x = \frac{1}{4}$

$x = 360°n \pm \cos^{-1}\frac{1}{4}$

$= 360°n \pm 76°$

Qn	Solutions	Marks	Comments: Criteria

c) $P(x) = x^3 + ax^2 + bx - 18$

$P(-2) = 0$

$P(1) = -24$

$-8 + 4a - 2b - 18 = 0$

$1 + a + b - 18 = -24$

$4a - 2b = 26$

$a + b = -7 \quad \times 2$

$4a - 2b = 26 \quad +$

$2a + 2b = -14$

$6a = 12 \qquad b = -7 - a$

$a = 2 \qquad b = -9$

d) $4\cos x + 3\sin x = R\sin(x + \alpha)$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = 3 \qquad \cos\alpha = \frac{3}{5} > 0$ 1st quadr.

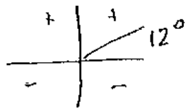
$R\sin\alpha = 4 \qquad \sin\alpha = \frac{4}{5} > 0$

$R^2\cos^2\alpha = 9 \qquad \alpha \approx 53°$

$+ \; R^2\sin^2\alpha = 16$

$R^2 = 25 \quad \therefore R = 5$

$4\cos x + 3\sin x = 5\sin(x + 53°)$



$5\sin(x + 53°) = -1$

$\sin(x + 53°) = -\frac{1}{5}$

$x + 53° = 180° + 12°,\ 360° - 12°$

$= 192°,\ 348°$

$x = 139,\ 295$

Qn	Solutions	Marks	Comments: Criteria

5 a) $P(x) = x^3 - 2x^2 - 5x + 6$

(i) $P(1) = 1 - 2 - 5 + 6$

$= 0 \quad \therefore x - 1$ is a factor of $P(x)$

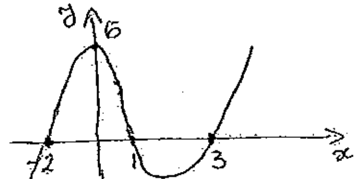
(ii)

$$\begin{array}{r} x^2 - x - 6 \\ x-1 \,\big)\, \overline{x^3 - 2x^2 - 5x + 6} \\ -\; x^3 - x^2 \qquad\qquad \\ \hline -x^2 - 5x + 6 \\ -\; -x^2 + x \qquad \\ \hline -6x + 6 \\ -6x + 6 \\ \hline 0 \end{array}$$

$P(x) = (x-1)(x^2 - x - 6)$

$= (x-1)(x-3)(x+2)$

(iii)



(iv) $\frac{(x-3)(x+2)}{(x-1)} \geq 0 \quad$ for $-2 \leq x < 1$ or $x \geq 3$

b) $\frac{1}{1\times 4} + \frac{1}{4\times 7} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}, \; n \geq 1$

1) Prove for $k = 1$

$\frac{1}{1\times 4} = \frac{1}{3\times 1 + 1}$

$\frac{1}{4} = \frac{1}{4}$

The statement is true for $k=1$

2) Assume that it's true for $n = k$ and prove for $n = k+1$

$\frac{1}{1\times 4} + \frac{1}{4\times 7} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

prove:

$\underbrace{\frac{1}{1\times 4} + \frac{1}{4\times 7} + \dots + \frac{1}{(3k-2)(3k+1)}}_{\frac{k}{3k+1}} + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$ from the assumption

Qn	Solutions	Marks	Comments: Criteria
	LHS: $\frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$		
	$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$		
	$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$		
	$= \frac{(k+1)\cancel{(3k+1)}}{(3k+4)\cancel{(3k+1)}}$		
	$= \frac{k+1}{3k+4}$		
	The statement is true for $n=k+1$		
	3) From (1) and (2) by Math. induction the statement is true for all $n \geq 1$		
c)	$4x^3+32x^2+79x+60=0$		
	$\alpha, \beta, \alpha+\beta$		
	$\alpha+\beta+\alpha+\beta = -\frac{32}{4}$		
	$2\alpha+2\beta=-8$		
	$2(\alpha+\beta)=-8 \therefore \boxed{\alpha+\beta=-4}$		
	$\alpha\beta(\alpha+\beta) = -\frac{60}{4}$		
	$\alpha\beta\times(-4) = -15$		
	$4\alpha\beta=15$		
	$\alpha+\beta=-4$ (1) $\therefore \alpha=-4-\beta$		
	$4\alpha\beta=15$ (2)		
	$4(-4-\beta)\beta-15=0$		
	$-16\beta - 4\beta^2-15=0$		
	$4\beta^2+16\beta+15=0$		
	$\beta=\frac{-16\pm\sqrt{256-240}}{8} = \frac{-16\pm4}{8}$ $\begin{cases} -\frac{20}{8} = -\frac{5}{2} \\ -\frac{12}{8} = -\frac{3}{2}\end{cases}$		
	$\beta=-\frac{3}{2}$ $\alpha=-\frac{5}{2}$		

Qn	Solutions	Marks	Comments: Criteria
6 a)	$\tan\alpha=\frac{1}{3}$		
	$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}}$		
	$= \frac{3}{4}$		
	$\tan(2\alpha+\beta) = \frac{\tan 2\alpha+\tan\beta}{1-\tan 2\alpha\tan\beta}$		
	$= \frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}$		
	$= 1$		
	$\therefore 2\alpha+\beta=45^\circ$		
b)	LHS: $\frac{\cot\theta}{1-\operatorname{cosec}\theta} = \frac{\frac{1-t^2}{2t}}{1-\frac{1+t^2}{2t}}$		
	$= \frac{1-t^2}{2t-1-t^2}$		
	$= \frac{(1-t)(1+t)}{-(t-1)^2}$		
	$= \frac{\cancel{(1-t)}(1+t)}{-(1-t)^{\cancel{2}}}$		
	$= \frac{1+t}{t-1}$		
	$=$ RHS		
c)	7^n+6^n is divisible by 13 for all odd positive integers.		
	1) Prove for $n=1$		
	$7+6=13$		
	13 is div. by 13		
	The statement is true for $n=1$		

Qn	Solutions	Marks	Comments: Criteria
	2) Assume it's true for $n=k$		
	7^k+6^k is divisible by 13		
	$\therefore 7^k+6^k = 13m \quad \therefore 7^k = 13m-6^k$		
	Prove for $n=k+2$		
	$7^{k+2}+6^{k+2} = 7^2 \times 7^k + 6^2 6^k$		
	$= 49\times 7^k + 36\times 6^k$		
	$= 49(13m-6^k)+36\times 6^k$		
	$= 49\times 13m - 49\times 6^k + 36\times 6^k$		
	$= 49\times 13m - 13\times 6^k$		
	$= 13(49m-6^k)$		
	$= 13P$ is div. by 13		
	The statement is true for $n=k+2$		
	3) It follows from (1) and (2) by MI that the statement is true for all odd positive integers.		
d)	$P(x) = x^n(2^m-1)+x^m(1-2^n)+2^n-2^m$		
	$P(1) = 1\times(2^m-1)+1\times(1-2^n)+2^n-2^m$		
	$2^m-1+1-2^n+2^n-2^m$		
	$=0$		
	$\therefore (x-1)$ is a factor of $P(x)$		
	$P(2) = 2^n(2^m-1)+2^m(1-2^n)+2^n-2^m$		
	$= 2^{n+m}-2^n+2^m-2^{m+n}+2^n-2^m$		
	$=0$		
	$\therefore (x-2)$ is a factor of $P(x)$		
	$\therefore (x-1)(x-2)$ is a factor of $P(x)$		