BRIGIDINE RANDWICK

FORTITER ET SUAVITER

Mathematics
HSC Assessment
June 7
Task 3 2017

General Instructions

- Time Allowed – 50 minutes
- Write using blue or black pen only
- Draw any relevant diagrams using pencil
- Board-approved calculators may be used
- All necessary working should be shown in every question

Total marks **(40)**

Multiple Choice

1. The function $f(x) = -3\cos\left(\frac{\pi x}{5}\right)$ has a period of

A. $\frac{\pi}{5}$ B. $\frac{\pi}{10}$ C. 3 D. 10

2. A particle is moving in a straight line. It's distance (x metres) from a fixed point O is given by $x = 2\sin 2t$, where t is the time in seconds.

At which times is the particle at rest?

A. $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots\ldots$

B. $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots\ldots\ldots$

C. $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots\ldots\ldots$

D. $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots\ldots\ldots$

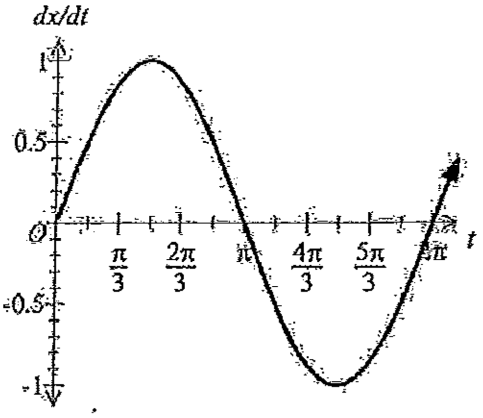
3. A circular metal plate of area $A\,cm^2$ is being heated. It is known that the rate of increase in area of the plate after a certain time "t" is given by

$$\frac{dA}{dt} = \frac{\pi t}{32}\ cm^2/h$$

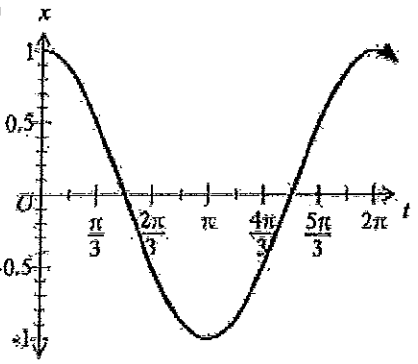
What is the exact area of the plate after 8 hours, if initially the plate had a radius of 6cm?

A. π B. 0.25π C. 36π D. 37π

4. Which of the following graphs could be the displacement function of the velocity function below?



(A)



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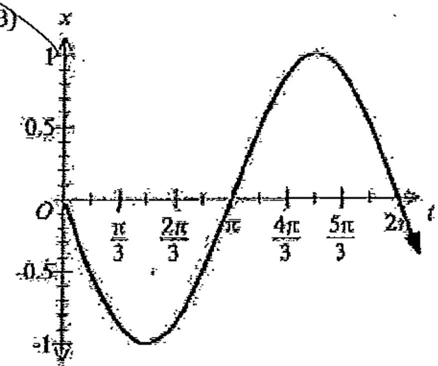

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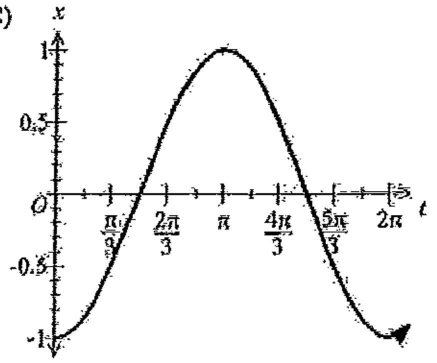
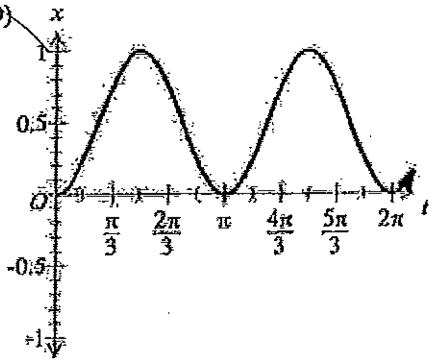


(D)



Question 5

Trigonometric Functions (17 Marks)

a. Differentiate the following with respect to x.

i. $y = \cos 3x$ 1

ii. $y = e^{2x} \tan 2x$ 2

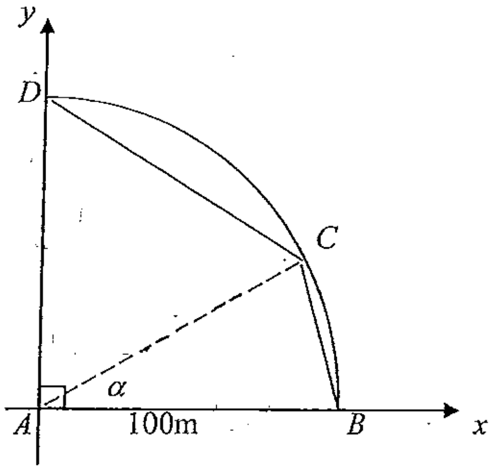
b. If $y = \cot x$ show that $\frac{dy}{dx} = -\operatorname{cosec}^2 x$ 3

c. Evaluate $\lim_{x \to 0} \frac{\sin 5x}{7x}$ (show all working) 1

d. Find $\int_0^{\frac{\pi}{6}} \frac{dx}{\cos^2 x}$ in exact form. 2

e. Find the volume generated when the curve $y = \sqrt{\cot x}$ is rotated about the x axis between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$. Leave your answer in exact simplified form. 3

f. $ABCD$ is a quadrilateral inscribed in a quarter of a circle centred at A with radius 100m. The points B and D lie on the x and y axes and the point C moves on the circle such that $\angle CAB = \alpha$ as shown in the diagram below.



i. Show that the area of the quadrilateral $ABCD$ can be expressed as **2**

$$A = 5000(\sin\alpha + \cos\alpha)$$

ii. Show that the maximum area of this quadrilateral is $5000\sqrt{2}\,m^2$. **3**

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Question 6 START A NEW PAGE

Applications of Calculus to the Physical World (19 Marks)

a. The mass M kg of radioactive substance present after t years is given by the equation

$$M = M_o e^{-kt}$$

where k is a positive constant.

After 50 years the substance has been reduced from 20kg to 10 kg in mass.

i. Clearly show that $\frac{dM}{dt} = -kM$. 1

ii. State the value of M_o. 1

iii. Show that the exact value of k is $\frac{\ln 2}{50}$ 2

iv. Find the time for the substance to lose $\frac{4}{5}$ of its original mass (to 1 dec. pl.) 2

b. The displacement of a particle is given by:

$$x = t - 4\log_e(t-1) + 5 \quad , \qquad t > 1$$

where x is in metres and t is in seconds.

i. Find the exact displacement of the particle when $t = 8$. 1

ii. Find an expression for v and hence find when the particle comes to rest. 2

iii. Clearly show that the acceleration remains positive for $t > 1$. 2

iv. Find the exact distance travelled by the particle between the times the particle comes to rest and $t = 8$. 2

c. A swimming pool is being emptied for maintenance. The quantity of water Q litres, remaining in the pool at any time, t minutes, after it starts to empty is given by:

$$Q(t) = 2000(25 - t)^2 \ , \ t \geq 0$$

i. At what rate is the pool being emptied at any time, t. 1

ii. How long will it take to half empty the pool to 1 decimal place? 2

iii. At what time is the water flowing out at 20kL/minute. 1

iv. What is the average water flow in the first 10 minutes in litres? 2

END OF ASSESSMENT TASK

Brigidine Mathematics

HSC Assessment Task 3 2017

SAMPLE SOLUTIONS

MC.

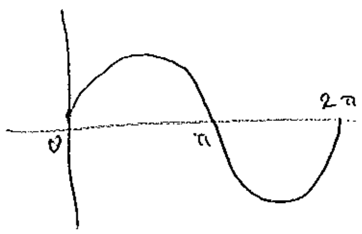
1. $f(x) = -3\cos\left(\frac{\pi x}{5}\right)$

Period $= \dfrac{2\pi}{\left(\frac{\pi}{5}\right)} = \dfrac{10\pi}{\pi} = 10$

(D)

2. $x = 2\sin 2t$.

0 at $2t = 0, \pi, 2\pi \ldots$



$\therefore$ at $t = 0, \frac{\pi}{2}, \pi$

we want at 0 velocity (at rest)

ie we differentiate

we get $4\cos 2t$

Cos t is at 0 when $t =$

$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

so $2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$\therefore t = \frac{\pi}{4}, \frac{3\pi}{4} \ldots$ $\Rightarrow$ (D)

3. $\dfrac{dA}{dt} = \dfrac{\pi t}{32}$.

$dA = \dfrac{\pi t}{32}\,dt$

$A = \displaystyle\int \frac{\pi t}{32}\,dt$

$= \dfrac{\pi t^2}{64} + C$

at $t = 0$

$r = 6$cm.

$\therefore A = \pi r^2 = 36\pi$

$A = \dfrac{\pi t^2}{64} + 36\pi$

at $t = 8$.

$A = \dfrac{\pi 64}{64} + 36\pi$

$= 37\pi \Rightarrow$ (D)

4. graph is $\sin x$.

So we integrate $\sin x$.

$= -\cos x$.

which is $\Rightarrow$ (C)

5. a. i) $y = \cos 3x$

$$\frac{dy}{dx} = -3\sin 3x.$$

ii) $y = e^{2x}\tan 2x$.

use product rule.

let $u = e^{2x}$

$v = \tan 2x$.

$$\frac{du}{dx} = 2e^{2x}.$$

$$\frac{dv}{dx} = 2\sec^2 2x.$$

$$\therefore \frac{dy}{dx} = 2e^{2x}\tan 2x + 2e^{2x}\sec^2 2x$$

$$= 2e^{2x}(\tan 2x + \sec^2 2x)$$

b. $y = \cot x. \Rightarrow = \frac{1}{\tan x} = \frac{\cos x}{\sin x}.$

use quotient rule.

$u = \cos x \quad v = \sin x$

$u' = -\sin x \quad v' = \cos x.$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{\sin x(-\sin x) - \cos^2 x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

$= -\operatorname{cosec}^2 x$

c. $\lim_{x \to 0} \frac{\sin 5x}{7x}$

$= \lim_{x \to 0} \frac{\sin 5x}{5x} \times \frac{5}{7}$

$1 \times 5/7$

$= 5/7$

d. $\int_0^{\pi/6} \frac{dx}{\cos^2 x}$

$= \int_0^{\pi/6} \sec^2 x \, dx$

$= \left[\tan x\right]_0^{\pi/6} \Rightarrow \tan \pi/6 - \tan 0$

$= \frac{1}{\sqrt{3}}$.

e. $y = \sqrt{\cot x}$

Volume $V = \pi \int_{\pi/4}^{\pi/3} y^2 dx$

$= \pi \int_{\pi/4}^{\pi/3} \cot x \, dx$.

$= \pi \int_{\pi/4}^{\pi/3} \frac{\cos x}{\sin x} dx$.

$= \pi \cdot \left[\ln(\sin x)\right]_{\pi/4}^{\pi/3}$

$\pi \left[\ln\left(\sin \frac{\pi}{3}\right) - \ln\left(\sin \frac{\pi}{4}\right)\right]$

$= \pi \left[\ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{\sqrt{2}}\right)\right]$

$= \pi \ln\left(\frac{\sqrt{3}/2}{1/\sqrt{2}}\right)$

$= \pi\left(\ln\left(\frac{\sqrt{6}}{2}\right)\right)$

f. Consider the areas of Δ ACD and Δ ACB

then r of circle = 100m

$\frac{1}{2}(10000) \sin(90 - \alpha)$

$+ \frac{1}{2}(10000) \sin(\alpha)$

$= 5000(\sin\alpha + \cos\alpha)$

ii) $\frac{dA}{d\alpha} = 5000\cos\alpha - 5000\sin\alpha$

$= 5000(\cos\alpha - \sin\alpha) = 0$

$\therefore \cos\alpha = \sin\alpha \Rightarrow$ at $\alpha = \pi/4$.

$\therefore A_{max} = \left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right) \times 5000$

$= 5000\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$= 5000\left(\frac{2}{\sqrt{2}}\right) = \left(\frac{2\sqrt{2}}{2}\right) 5000$

$= \sqrt{2}(5000)\, m^2$

6. $M = M_0 e^{-kt}$.

at $t = 0$, $M = 20$.

ie $M_0 = 20$.

at $t = 50$, $M = 10$

$20e^{-k50} = 10$

$e^{-k50} = \frac{10}{20}$

$-k50 = \ln\left(\frac{1}{2}\right)$

$k = \frac{-\ln\left(\frac{1}{2}\right)}{50}$

$\simeq 0.01386$

$\to k = \frac{-\ln(2^{-1})}{50}$

$= \frac{\ln(2)}{50}$.

i) $M = M_0 e^{-kt}$.

$\frac{dM}{dt} = -kM_0 e^{-kt}$.

$= -kM$.

ii) $M_0 = 20$

iii) $k = \frac{\ln(2)}{50}$ (see left).

iv) $\frac{4}{5} \times 20 = 16$ kg

$16 = 20e^{-kt}$.

$\ln\left(\frac{16}{20}\right) = -kt$.

$\frac{\ln\left(\frac{16}{20}\right)}{-\left(\frac{\ln(2)}{50}\right)} = t \simeq 16$ years.

$x = t - 4\ln(t-1) + 5$. $t > 1$.

at $t = 8$.

$x = 8 - 4\ln(7) + 5$.

$= 13 - 4\ln(7)$

ii). $v = \frac{dx}{dt} = 1 - 4\left(\frac{1}{t-1}\right)$

$= 1 - \frac{4}{t-1}$.

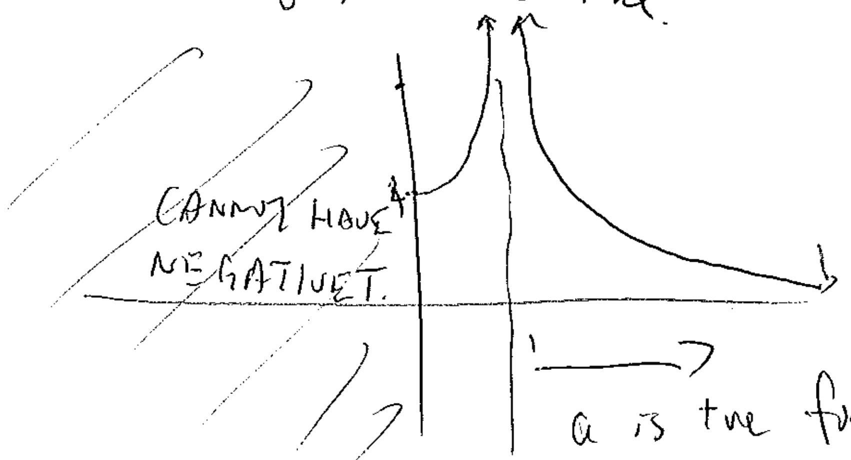
iii). $v = -4(t-1)^{-1}$

$\frac{dv}{dt} = 4(t-1)^{-2}$

$= \frac{4}{(t-1)^2} > 0,\ t \in \mathbb{R}$

$\therefore$ Acceleration remains +ve for $t > 1$

graph looks like.



a is +ve for all $t > 1$.

iv) rest at

$v = 0$,

$1 - \frac{4}{t-1} = 0$

$t - 1 = 4$

$t = 5$

$\int_5^8 1 - \frac{4}{t-1}\, dt$

$= \left[t - 4\ln(t-1)\right]_5^8$

$= (8 - 4\ln 7) - (5 - 4\ln 4)$

$= 3 - 4\ln 7 + 4\ln 4$

$= 3 + 4(\ln 4 - \ln 7)$

$= 3 + 4\left(\ln\frac{4}{7}\right)$

$Q(t) = 2000(25-t)^2,\ t \geq 0$

i) rate of emptied

$= \frac{dQ(t)}{dt} = \frac{d}{dt} 2000(25-t)^2$

$= \frac{d}{dt}\ 2000(2)(25-t)(-1)$

$= -4000(25-t)$

ii) the pool is obviously full at $t=0$.

So $Q(0) = 2000(25)^2$

$= 1250000L$

we want half of that, so 625000.

$625000 = 2000(25-t)^2$

$312.5 = (25-t)^2$.

$17.68 = 25-t$.

$t = 25 - 17.68$

$\approx 7.32\,min$

$\approx 7.3\,min$ (1dp.)

iii) $-20000 = -4000(25-t)$

$5 = 25 - t$.

$t = 20\,min$.

iv) $\frac{d}{dt} -4000(25-t)$

$= 4000$

$4000 \times 10 = 40000\ L/min$