KAMBALA

Student Number: ______ ______

HSC Task 2
June 2017

Extension 1 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour 30 minutes
- Write using black pen
- Board-approved calculators may be used
- Answer Questions 1 – 8 on the multiple choice answer sheet provided
- Answer Questions 9 – 12 on the paper provided
  Start each Question on a new page.
- A reference sheet is provided at the back of this paper
- In Questions 9 – 12, show relevant mathematical reasoning and/or calculations.

Total marks – 56

Section I
8 marks

- Attempt Questions 1 – 8
- Allow about 12 minutes for this section

Section II
48 marks

- Attempt Questions 9 – 12
- Allow about 1 hour and 18 minutes for this section

Section I

8 Marks
Attempt Questions 1 – 8
Allow about 12 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 8.

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1 Use the substitution $u=2x+1$, to find an expression for $\int(2x+1)^{10}dx$.

(A) $\frac{1}{22}(2x+1)^{11}+c$

(B) $\frac{1}{11}(2x+1)^{11}+c$

(C) $\frac{1}{18}(2x+1)^{9}+c$

(D) $\frac{1}{9}(2x+1)^{9}+c$

2 If $f(x)=e^{x+2}$ what is the equation of the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x)=e^{y-2}$

(B) $f^{-1}(x)=e^{y+2}$

(C) $f^{-1}(x)=\log_e x-2$

(D) $f^{-1}(x)=\log_e x+2$

3 Which of the following is an expression for $\int\cos^2 2x\,dx$?

(A) $x-\frac{1}{4}\sin 4x+c$

(B) $x+\frac{1}{4}\sin 4x+c$

(C) $\frac{x}{2}-\frac{1}{8}\sin 4x+c$

(D) $\frac{x}{2}+\frac{1}{8}\sin 4x+c$

4 The exact value of $\tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$ is

(A) $\frac{\sqrt{5}}{2}$

(B) $-\frac{\sqrt{5}}{2}$

(C) $\frac{2}{\sqrt{5}}$

(D) $-\frac{2}{\sqrt{5}}$

5 Given that $\cos x = \frac{4}{5}$ and $0 \le x \le \frac{\pi}{2}$, find the exact value of $\cos 2x$.

(A) $\frac{16}{25}$

(B) $\frac{7}{25}$

(C) $\frac{24}{25}$

(D) $\frac{8}{5}$

6 What is the domain of $y = \cos^{-1}(1 - 3x)$?

(A) $0 \le x \le \frac{\pi}{2}$

(B) $0 \le x \le \pi$

(C) $-\frac{2}{3} \le x \le 0$

(D) $0 \le x \le \frac{2}{3}$

7 A cubic polynomial is monic. Its roots are $\alpha, -\alpha$ and β. The sum of its roots is 4 and the product of its roots is -36. Which of the following represents the polynomial?

(A) $P(x)=x^3-4x^2-9x+36$

(B) $P(x)=x^3+4x^2-9x-36$

(C) $P(x)=2x^3-8x^2-18x+72$

(D) $P(x)=2x^3+8x^2-18x-72$

8 The parametric equations of a curve are $x=p+1$ and $y=p^2-1$. Which of the following is the Cartesian equation of the curve?

(A) $x^2=4y$

(B) $y=(x-2)^2$

(C) $y=x^2-2x$

(D) $y=x^2-2$

Section II

48 Marks
Attempt Questions 9 – 12
Allow about 1 hour 18 minutes for this section

Answer each question on the writing paper provided. *Start each question on a new page.*

In Questions 9 – 12, your responses should include relevant mathematical reasoning and/or calculations.

Question 9 (12 marks) Start a new page.

(a) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$. **1**

(b) Divide the interval AB externally in the ratio $4:3$ where A is the point $(2,-1)$ and B is the point $(1,-3)$. **2**

(c) Find the acute angle between the lines $2x - y + 1 = 0$ and $x + 3y - 4 = 0$. Write your answer to the nearest degree. **3**

(d) Find $\int \sin^2 \dfrac{x}{2}\, dx$. **2**

(e) Prove that $\dfrac{1-\cos 2A}{\sin 2A} = \tan A$ if $\sin 2A \neq 0$. **2**

(f) Write down the general solution, in terms of π, of the equation $\cos\theta = -\dfrac{1}{2}$. **2**

End of Question 9

Question 10 (12 marks) Start a new page.

(a) (i) Differentiate $x\sin^{-1}x+\sqrt{1-x^2}$. **2**

(ii) Hence, find the exact value of $\int_0^1 \sin^{-1}x\,dx$. **2**

(b) Find $\int 3^x\,dx$. **1**

(c) Using the substitution $u=x+2$, evaluate $\int_{-1}^{2} \frac{x}{3}\sqrt{x+2}\,dx$. **3**

(d) Consider the function $f(x)=\sec x$ for $0\le x\le\frac{\pi}{2}$.

(i) State the domain of the inverse function $f^{-1}(x)$. **1**

(ii) Show that $f^{-1}(x)=\cos^{-1}\left(\frac{1}{x}\right)$. **1**

(iii) Hence find $\frac{d}{dx}\left[f^{-1}(x)\right]$. **2**

End of Question 10

Question 11 (12 marks) Start a new page.

(a) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where $0 < \alpha < \dfrac{\pi}{2}$ and $R > 0$. **2**

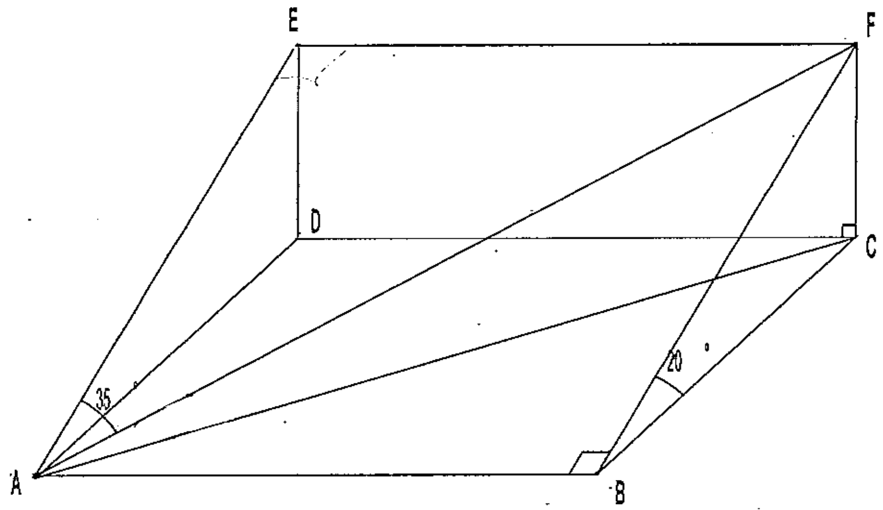
(ii) Hence, solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le \dfrac{\pi}{2}$. **1**

(b) For the function $f(x) = \sin x - \cos^2 x$

(i) Show that $f(x)$ has a root between $x = 2$ and $x = 3$. **2**

(ii) Starting with $x_1 = 2.2$ use one application of Newton's method to find a better approximation for the root. Answer correct to 2 significant figures. **2**

(c)



The diagram shows a hill inclined at $20°$ to the horizontal. A straight road AF on the hill makes an angle of $35°$ with the line of greatest slope. (Let CF = DE = h)

(i) Show that $\text{BF} = \dfrac{h}{\sin 20°}$. **1**

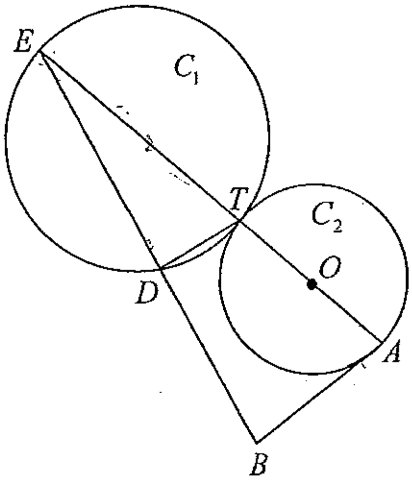
(ii) Show that $\text{AF} = \dfrac{h}{\sin 20° \cos 35°}$. **2**

(iii) Hence, find, correct to the nearest degree, $\angle\text{CAF}$ the inclination of the road AF to the horizontal. **2**

End of Question 11

Question 12 (12 marks) Start a new page.

(a)



Not to scale

Two circles C_1 and C_2 touch at T. O is the centre of C_2. AE and BD are straight lines. BA is a tangent to C_2. The radius of C_1 is R and the radius of C_2 is r.

(i) Find the size of $\angle EDT$, giving reasons. **2**

(ii) If $DE = 2r$, find an expression for the length of EB in terms of R and r. **2**

(b) Use mathematical induction to prove that **3**

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

for all positive integers n.

(c) (i) Find $\frac{d}{dx}\left[\tan^{-1}\left(\frac{1}{x}\right)\right]$. **1**

(ii) Hence find the value of $y=\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)$ for $x>0$. **2**

(iii) Draw the graph of $y=\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)$ and determine if it is continuous. **2**

End of Paper

KAMBALA

HSC TASK 2 EXT 1

SAMPLE SOLUTIONS

1. $\int (2x+1)^{10} dx$

let $u = 2x+1$

$du = 2dx \Rightarrow dx = \frac{du}{2}$

$\int (2x+1)^{10} dx \equiv \int u^{10} \frac{du}{2}$

$= \frac{1}{2} \int u^{10} du$

$= \frac{1}{2}\left(\frac{u^{11}}{11}\right) + C$

$= \frac{(2x+1)^{11}}{22} + C \Rightarrow$ A

2. $f(x) = e^{x+2}$.

$\ln(f(x)) = x+2$

$x = \ln(f(x)) - 2$

$f^{-1}(x) = \ln(x) - 2 \Rightarrow$ C

3. $\int \cos^2 2x \, dx$.

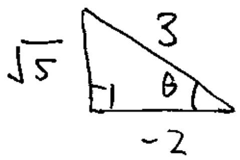
$= \int \frac{1}{2}(1 + \cos 4x) \, dx$

$= \frac{1}{2}\left[x + \frac{\sin 4x}{4}\right]$

$= \frac{x}{2} + \frac{1}{8}\sin 4x + C$

$\Rightarrow$ D

4. $\tan(\cos^{-1}(-\frac{2}{3}))$

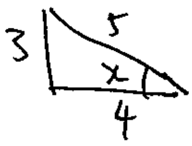


$\tan\theta = \frac{\sqrt{5}}{-2}$.

$\Rightarrow$ B.

5. $\cos x = \frac{4}{5}$ $\quad 0 \le x \le \frac{\pi}{2}$.

$\cos 2x = \cos^2 x - \sin^2 x$.



$\cos x = \frac{4}{5}$

$\sin x = \frac{3}{5}$

$\left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25}$

$= \frac{7}{25} \Rightarrow$ B

6. $y = \cos^{-1}(1-3x)$

$\cos(y) = 1-3x.$

$-1 < 1-3x < 1$

$0 \le x \le 2/3$

$\Rightarrow$ D

7. ax^3+bx^2+cx+d

$a=1$

$\alpha, -\alpha, \beta.$

Sum of roots $= -b/a = -b = \beta = 4.$

$b = -4$ $\Rightarrow$ A

8. $x = p+1 \rightarrow p = x-1$

$y = p^2-1.$

$y = (x-1)^2-1$

$y = x^2+1-2x-1$

$y = x^2-2x.$

$\Rightarrow$ C

9. a) $\lim_{x \to 0} \frac{\sin 3x}{x} \times \frac{3}{3}$

$= 3 \times \frac{\sin 3x}{3x}$

$= 3 \times 1 = 3$

b) $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n} \right)$

$\left(\frac{4(1)-3(2)}{4-3}, \frac{4(-3)-3(-1)}{4-3} \right)$

$= \left(\frac{4-6}{1}, \frac{-12+3}{1} \right)$

$= (-2,-9)$

c) $y = 2x+1 \rightarrow m = 2.$

$y = \frac{-x+4}{3} \rightarrow m = -1/3.$

$\tan\theta = \left| \frac{m_1-m_2}{1+m_1m_2} \right|$

$= \left| \frac{2+\frac{1}{3}}{1+\left(-\frac{2}{3}\right)} \right| = \left| \frac{\frac{7}{3}}{\frac{1}{3}} \right| = 7$

$\tan^{-1}(7) \approx 82^\circ$ (Nearest degree)

d) $\int \sin^2 \frac{x}{2}\,dx$

$= \int \frac{1}{2}(1-\cos x)\,dx$

$= \frac{1}{2}\int 1 - \cos x\,dx$

$= \frac{1}{2}\left[x - \sin x\right] + C$

e) $\frac{1-\cos 2A}{\sin 2A} = \tan A.$

$\sin^2\theta + \cos^2\theta = 1$

$1 - \cos^2\theta = \sin^2\theta$

$\frac{1-(\cos^2 A - \sin^2 A)}{2\sin A\cos A}$

$= \frac{\sin^2 A + \sin^2 A}{2\sin A\cos A}$

$= \frac{2\sin^2 A}{2\sin A\cos A} = \frac{\sin A}{\cos A} = \tan A$

given that $\sin 2A \neq 0$

f) $\cos\theta = -\frac{1}{2}. \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{2\pi}{3}, \frac{4\pi}{3} \ldots$

$\theta = 2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

$= 2n\pi \pm \frac{2\pi}{3}$

10. $x\sin^{-1}x + \sqrt{1-x^2}.$

$= 1(\sin^{-1}x) + \frac{1}{\sqrt{1-x^2}}x.$ ①

$= \frac{d}{dx}\, x\sin^{-1}x$ [product rule]

$\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{2\sqrt{1-x^2}} x - 2x.$

$= -\frac{x}{\sqrt{1-x^2}.}$ ②

$\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right) = ① + ②$

$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x.$

ii) $\int_0^1 \sin^{-1}x\,dx$

$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= \frac{\pi}{2} - 1$

b) $\int 3^x\,dx = \frac{3^x}{\ln 3} + C$

c) let $u = x+2$ in

$$\int_{-1}^{2} \frac{x}{3}\sqrt{x+2}\,dx.$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx.$$

$$\int_{1}^{4} \frac{u-2}{3}\sqrt{u}\,du.$$

$$= \int_{1}^{4} \frac{u\sqrt{u} - 2\sqrt{u}}{3}\,du$$

$$= \frac{1}{3}\int_{1}^{4} u\sqrt{u} - 2\sqrt{u}\,du$$

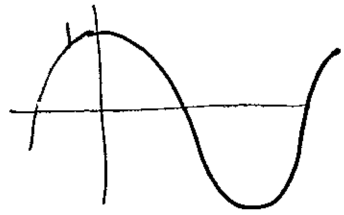
$$= \frac{1}{3}\left[\frac{u^{5/2}\times 2}{5} - \frac{2u^{3/2}\times 2}{3}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3}\right]$$

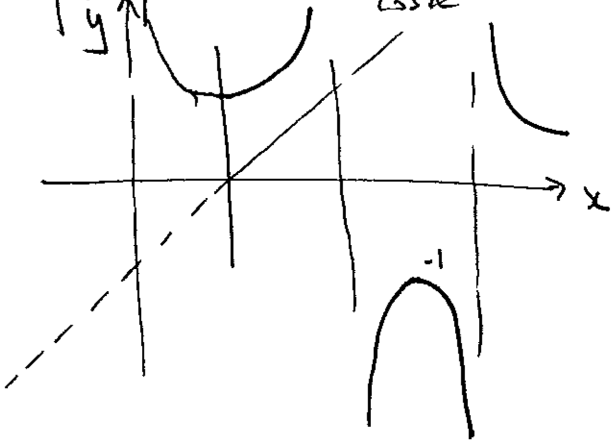
$$= \frac{46}{45}$$

d) $f(x) = \sec x.$

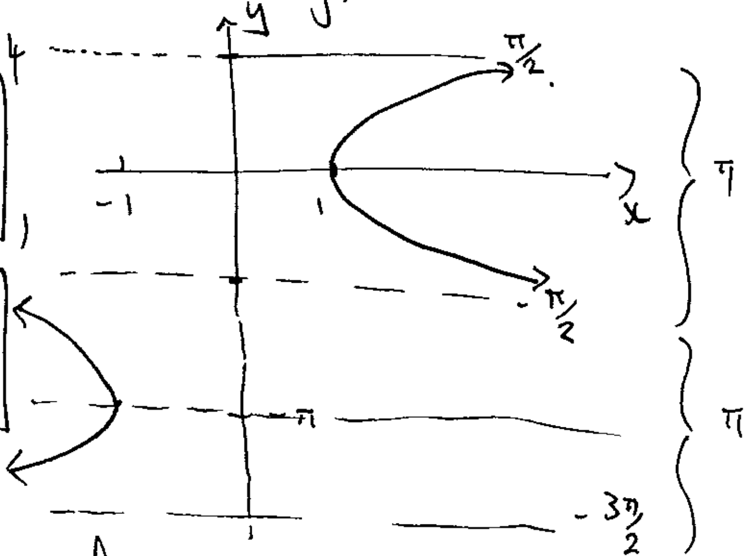
$\cos x$



$f(x) = \sec x = \frac{1}{\cos x}$



$x = \sec(y)$



in $f(x) \rightarrow$ domain $= x \in \mathbb{R}$.

$y \geq 1,\ y \leq -1$

in $f^{-1}(x) \rightarrow$ domain $= x \geq 1,\ x \leq -1$

range $= 0 \leq y \leq \pi;$

$y \neq \frac{\pi}{2}$

ii) $x = \sec(y)$

$x = \frac{1}{\cos(y)}$

$x\cos(y) = 1$

$\cos(y) = \frac{1}{x}$

$y = \cos^{-1}\left(\frac{1}{x}\right)$

iii) $\frac{d}{dx}\left[f^{-1}(x)\right]$

$= \frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right).$

$= \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$

$= \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right)$

$= \frac{1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}}$

11. $\sqrt{3}\cos x - \sin x.$

$R(\cos x\cos\alpha - \sin x\sin\alpha).$

$R\cos\alpha = \sqrt{3}$, $R\sin\alpha = 1.$

$\left(\frac{\sqrt{3}}{R}\right)^2 + \left(\frac{1}{R}\right)^2 = 1$

$\frac{3}{R^2} + \frac{1}{R^2} = 1$

$R^2 = 4$, $R = 2.$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$

$\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \pi/6$

$\therefore = 2\cos(x + \pi/6)$ in the form of $R\cos(x+\alpha).$

ii) $2\cos(x + \pi/6) = 1$

$\cos(x + \pi/6) = 1/2.$

$x + \pi/6 = \pi/3.$

$x = \pi/6$

b) $f(x) = \sin x - \cos^2 x$.

at $x = 2$.

$f(x) = \sin x - \cos^2 x. = +ve$.

$f(3) = \sin 3 - \cos^2 3 = -ve$

because the function is continuous

there must be a root between $2 < x < 3$.

ii) $f(x) = \sin x - \cos^2 x$.

$= \sin x - \frac{1}{2}(1 + \cos 2x)$

$f'(x) = \cos x + \frac{2\sin 2x}{2}$.

$= \cos x + \sin 2x$.

$f(2.2) \approx 0.462$.

$f'(2.2) \approx -1.54$.

$= 2.2 - \left(\frac{0.462}{-1.54}\right)$

$= 2.5$ (2 s.f.)

c) in ΔBCF

$CF = h$

$\frac{CF}{BF} = \sin 20$

$\frac{h}{BF} = \sin 20$

$BF = \frac{h}{\sin 20}$.

ii). in ABFE

$\angle AEF = \angle AFB = 35°$

(Alternate)

$\frac{BF}{AF} = \cos \angle AFB$

$\frac{\left(\frac{h}{\sin 20}\right)}{AF} = \cos 35$

$AF = \frac{h}{\sin 20 \cos 35}$

iii) $\frac{h}{AF} = \sin(\angle CAF)$

$\frac{h}{\left(\frac{h}{\sin 20 \cos 35}\right)} = \sin(\angle CAF)$

$\frac{h}{1} \times \frac{\sin 20 \cos 35}{h} = \sin(\angle CAF)$

$\sin^{-1}(\sin 20 \cos 35) = \angle CAF$

$\angle CAF = 16°$ (Nearest degree

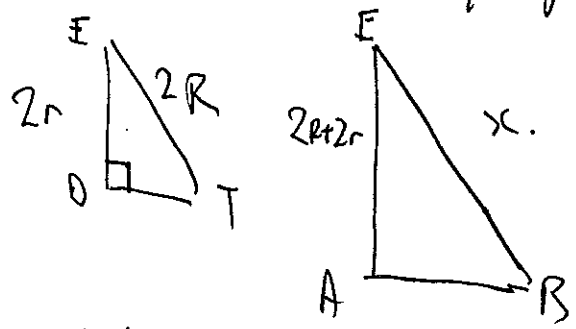
12. a) $\angle EAB = 90°$ (tangent meets radius at right angles)

ET is the diameter of C_1
(A line passing through the intercept of 2 circles passes through the centre of both circles)

$\therefore \angle EOT = 90°$ ($\angle$ in a semicircle is a right angle)

ii) we have 2 similar triangles

$\Delta EOT \,|||\, \Delta EAB$ (equiangular)



let length EB be x.

$$\frac{2R+2r}{2r} = \frac{x}{2R}$$

$$2R(2R+2r) = x(2r)$$

$$x = \frac{4R^2+4Rr}{2r} =$$

$$\frac{4(R^2+Rr)}{2r} = \frac{2(R^2+Rr)}{r}$$

b) $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

prove true for $n=1$

$$\frac{1}{1(1+1)}=\frac{1}{1+1} \checkmark \quad \left(\frac{1}{2}=\frac{1}{2}\right)$$

true for $n=1$

Assume true for $n=k$, where $0\leq k\leq n$

$$\underbrace{\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots\frac{1}{k(k+1)}}=\frac{k}{k+1} \quad ①$$

Prove true for $n=k+1$

$$\underbrace{\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots\frac{1}{k(k+1)}}_{=\frac{k}{k+1}}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}. \quad ②$$

$$\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

$$\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}=②$$

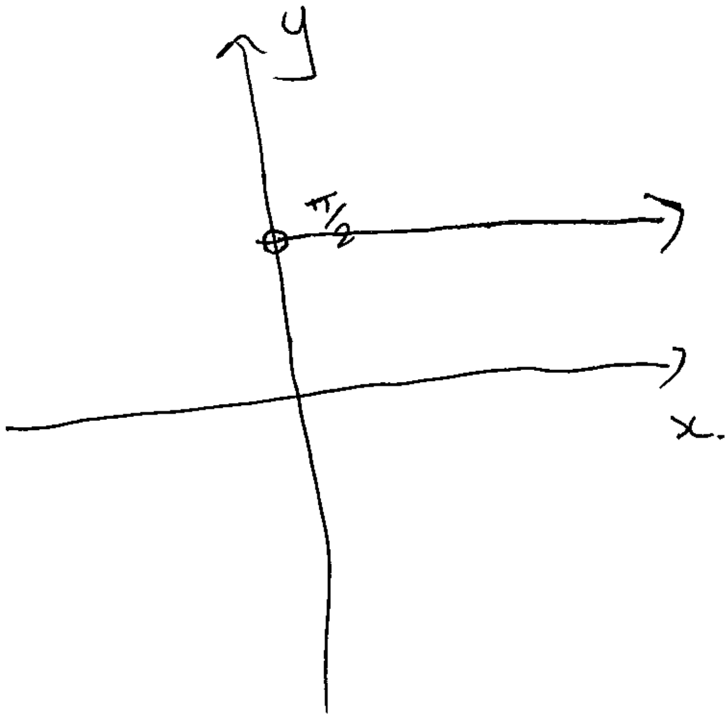
Since its true for $n=1$ and $n=k+1$ when true for $n=k$, this equation is true for all positive integers n

c) $\frac{d}{dx}\left[\tan^{-1}\left(\frac{1}{x}\right)\right]$

$= \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \frac{-1}{x^2}$

$= \frac{1}{1+\left(\frac{1}{x^2}\right)} \times \frac{-1}{x^2}$

$= \frac{-1}{x^2+1}$



the function is not continuous because it is undefined at $x=0$.

$y = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$

at $x=1$

$\tan^{-1}(1) + \tan^{-1}(1) = \frac{\pi}{2}$

$\therefore\ y = \pi/2$ for all $x>0$.