2017 HSC

YEAR 12

HSC HALF YEARLY EXAM

Student Name:

Section 1	
Section II	
Question 6	
Question 7	
Question 8	
TOTAL	

Mathematics Extension 1

General Instructions

- Reading time - 5 minutes
- Working time – 1.5 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- Show relevant mathematical reasoning and/or calculations in Questions 6-8

Total marks - 50

Section I
5 marks

- Attempt Questions 1-5
- Allow about 8 minutes for this section

Section II
45 marks

- Attempt Questions 6-8
- Allow about 1 hour 22 minutes for this section

Section I

5 marks
Attempt Questions 1 - 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1-5

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1 Which of the following is the correct expression for $\int \frac{dx}{\sqrt{4-x^2}}$?

(A) $\cos^{-1}\frac{x}{2}+C$

(B) $\cos^{-1}2x+C$

(C) $\sin^{-1}\frac{x}{2}+C$

(D) $\sin^{-1}2x+C$

2 A curve has parametric equations $x=t-3$ and $y=t^2+2$. What is the Cartesian equation of this curve?

(A) $y=x^2-x-1$

(B) $y=x^2+x-1$

(C) $y=x^2-6x+11$

(D) $y=x^2+6x+11$

3 Which of the following is the inverse function of $y=x^2-6x+8$?

(A) $y=3\pm\sqrt{x-1}$

(B) $y=3(x+1)$

(C) $y=\sqrt{x+1}$

(D) $y=3+\sqrt{x+1}$

4 What is the greatest coefficient in the expansion of $(5+2x)^{12}$?

(A) ${}^{12}C_3\times5^9\times2^3$

(B) ${}^{12}C_4\times5^8\times2^4$

(C) ${}^{12}C_5\times5^7\times2^5$

(D) ${}^{12}C_6\times5^9\times2^3$

5 Which of the following is the range of $y = 4\tan^{-1}\left(\frac{x}{2}\right)$?

(A) $\{y : 0 \le y \le 4\pi\}$

(B) $\{y : -2\pi < y < 2\pi\}$

(C) $\{y : -\pi < y < \pi\}$

(D) $\{y : -2\pi \le y \le 2\pi\}$

End of Section I

Section II

45 marks
Attempt Questions 6 – 8
Allow about 1 hour and 22 minutes for this section

Answer each question in the appropriate writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks) **Marks**

(a) Find $\int \frac{1}{\left(1+\sqrt{y}\right)\sqrt{y}}\,dy$ using the substitution $y = u^2$ $(u > 0)$. **2**

(b) (i) Show that the equation $f(x) = \sin x + \cos x - x$ has a root between $x = 1$ and $x = 2$. **2**

(ii) Let $x = 1.2$ be a first approximation to the root. Apply Newton's method once to obtain a better approximation to the root. Answer correct to 3 significant figures. **2**

(c) (i) Given $(3+4x)^{15}$, use the Binomial Theorem to write an expression for T_k, $0 \le k \le 15$. **1**

(ii) Show that $\frac{T_{k+1}}{T_k} = \frac{64-4k}{3k}$. **3**

(iii) Hence find the greatest co-efficient in the expansion of $(3+4x)^{15}$. **2**

(d) Use the substitution $u = 6 - x$ to evaluate $\int_1^6 x\sqrt{6-x}\,dx$. **3**

End of Question 6

Question 7 (15 marks) **Marks**

(a) What are the roots of the equation $x^3+6x^2-x-30=0$ given one root is the sum of the other two roots? **3**

(b) State the domain and range of $y=4\cos^{-1}\left(\frac{3x}{2}\right)$. **2**

(c) The probability that any one of the 30 days in June is raining is 0.3. Write an expression for the probability that June will have exactly 10 rainy days. **2**

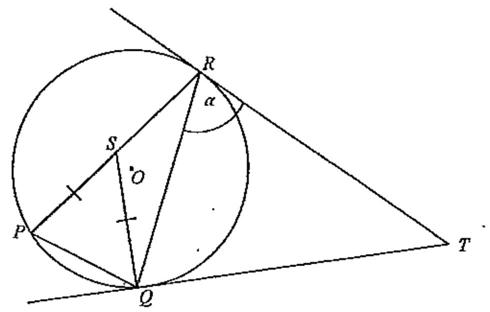
(d) $P(2p,p^2)$ and $Q(2q,q^2)$ are two points on the parabola $x^2=4y$.
M is the midpoint of PQ.

(i) Show that $(p-q)^2=2(p^2+q^2)-(p+q)^2$. **1**

(ii) If P and Q move on the parabola so that $p-q=4$, show that the locus of M is the parabola $x^2=4y-16$. **2**

(iii) What is the focus of the locus of M? **1**

(e) In the figure below TQ and TR are tangents to a circle centre O. P lies on the circle and QS=SP. $\angle QRT=\alpha°$



(i) Show that $\angle RSQ=2\alpha$. **2**

(ii) Hence show that $SQTR$ is a cyclic quadrilateral. **2**

End of Question 7

Question 8 (15 marks) **Marks**

(a) (i) Show that $\int_0^{\frac{\pi}{2}}\sin^2 x\,dx=\frac{\pi}{4}$. **2**

(ii) The area bounded by the curve $y=\sin x+1$, the axes and the line $x=\frac{\pi}{2}$ is rotated about the x-axis. **3**

Using the result from (i) or otherwise find the volume of this solid of revolution. Leave your answer in exact form.

(b) Consider the binomial expansion

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n$$

(i) Show that $1-{}^nC_1+{}^nC_2-\ldots+(-1)^n\,{}^nC_n=0$ **1**

(ii) Show that ${}^nC_1+2\,{}^nC_2+\ldots+n\,{}^nC_n=n2^{n-1}$ **2**

(c) A clay shooter hits the target 95% of the time. In a competition he will have forty shots at the target.

(i) What is the probability he hits 36 targets? Answer correct to 4 decimal places. **1**

(ii) What is the probability he misses at most two times? Answer correct to 4 decimal places. **2**

(d) Consider the function $y=\frac{1}{2}\cos^{-1}(x-1)$.

(ii) Sketch the graph of the function showing the coordinates of the endpoints. **1**

(iii) The region in the first quadrant bounded by the curve $y=\frac{1}{2}\cos^{-1}(x-1)$ and the coordinate axes is rotated through 360° about the y-axis. Find the volume of the solid of revolution. Express your answer in simplest exact form. **3**

End of paper

SSHS EXT 1 HALF YEARLY - SAMPLE SOLUTIONS

1. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

This is in the form of

$\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

where $a=2$

Answer $= \sin^{-1}\frac{x}{2} + C$.

$= C$

2. $x = t-3$

$y = t^2+2$

$t = x+3$

$y = (x+3)^2+2$

$y = x^2+9+6x+2$

$y = x^2+6x+11$

$= D$

3. $y = x^2-6x+8$

$x = y^2-6y+8$

$x = (y-3)^2-9+8$

$x = (y-3)^2-1$

$(y-3)^2 = x+1$

$y-3 = \sqrt{x+1} \Rightarrow$ Provided $x+1 \geq 0$

$y = 3+\sqrt{x+1}$

$= D$

4. Greatest Coefficient (1)

$= {}^{12}C_3 \times 5^9 \times 2^3$

$= A$

5. $y = 4\tan^{-1}\left(\frac{x}{2}\right)$

$\frac{y}{4} = \tan^{-1}\left(\frac{x}{2}\right)$

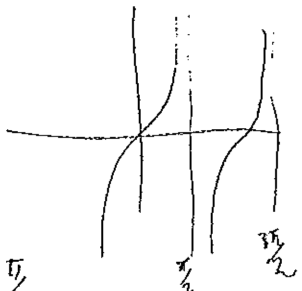
$\tan\left(\frac{y}{4}\right) = \frac{x}{2}$

$x = 2\tan\left(\frac{y}{4}\right)$

$-\frac{\pi}{2} < \frac{y}{4} < \frac{\pi}{2}$

$-2\pi < y < 2\pi$

$= B$



Question 6

$\displaystyle\int \frac{1}{(1+\sqrt{y})\sqrt{y}}\,dy$

let $y = u^2$

$\frac{dy}{du} = 2u \rightarrow dy = 2u\,du$.

$= \displaystyle\int \frac{1}{(1+u)(u)}\,2u\,du$

$= \displaystyle\int \frac{2}{1+u}\,du$

$= 2\displaystyle\int \frac{1}{1+u}\,du$

$= 2\ln(1+u)+C$

$= 2\ln(1+\sqrt{y})+C$

b) $f(x) = \sin x + \cos x - x$

at $x=1$ [ASSUMING RAD, NOT DEG]

$\sin(1)+\cos(1)-1 > 0$

$\sin(2)+\cos(2)-2 < 0$

Because $f(x)$ is continuous (no restrictions on $x \rightarrow x \in \mathbb{R}$)

There must lie a root between $x=1$ and $x=2$.

ii) $f'(x) = \cos x - \sin x - 1$ (2)

$f'(1.2) = -1.5697$

$f(1.2) = 0.0944$

$x_2 = 1.2 - \frac{f'(1.2)}{f(1.2)}$

$x_2 \approx 1.26$ (3sf.)

c) $T_{k+1} = {}^{15}C_k\,(3)^{15-k}(4x)^k$

$T_k = {}^{15}C_{k-1}\,(3)^{15-(k-1)}(4x)^{k-1}$

$= {}^{15}C_{k-1}\,3^{16-k}(4x)^{k-1}$

$$\frac{T_{k+1}}{T_k} = \frac{n-k+1}{k} \times \frac{b}{a}$$

$$= \frac{15-k+1}{k} \times \frac{4}{3}$$

$$= \frac{4(16-k)}{3k}$$

$$= \frac{64-4k}{3k}$$

iii) Greatest Coefficient at

$$\frac{T_{k+1}}{T_k} \geq 1$$

$$\frac{64-4k}{3k} \geq 1$$

$$64-4k \geq 3k$$

$$64 \geq 7k$$

$$k \leq 9.14$$

$$k = 9.$$

Greatest Coefficient at T_{10}

$$= {}^{15}C_9 (3)^{15-9}(4)^9$$

$$= {}^{15}C_9\, 3^6\, 4^9$$

d) $u = 6-x. \rightarrow \frac{du}{dx} = -1$

$$\int_1^6 x\sqrt{6-x}\,dx$$ 3

$$= \int_5^0 (6-u)\sqrt{u}\,(-du)$$

$$= \int_5^0 (u-6)\sqrt{u}\,du.$$

$$= \int_5^0 u^{3/2} - 6u^{1/2}\,du$$

$$= \left[\frac{u^{5/2}\times 2}{5}\right]_5^0 - \left[\frac{6u^{3/2}\times 2}{3}\right]_5^0$$

$$= \left(-\frac{5^{5/2}\times 2}{5}\right) + \left(\frac{6(5^{3/2})\times 2}{3}\right)$$

$$= -22.36 + 44.72$$

$$\approx 22.36 \text{ (2dp.)}$$

$$x^3+6x^2-x-30=0$$

let roots be α, β, γ

let $\alpha = \beta+\gamma$.

So roots are $\beta+\gamma, \beta, \gamma$.

Sum of roots $= -b/a$

$$2\beta+2\gamma = -6 \Rightarrow \beta+\gamma = -3.$$

$$(\beta+\gamma)(\beta)(\gamma) = 30$$

$$\beta\gamma(\beta+\gamma) = 30$$

$$\beta^2\gamma + \beta\gamma^2 = 30$$

$$\beta\gamma(-3) = 30$$

$$\beta\gamma = -10$$

$$\beta+\gamma = -3.$$

$$\beta = -3-\gamma$$

$$\gamma(-3-\gamma) = -10$$

$$-3\gamma - \gamma^2 = -10$$

$$\gamma^2+3\gamma-10 = 0$$

$$(\gamma+5)(\gamma-2) = 0.$$

$$\gamma = 2, -5$$

$$\beta = -5, 2.$$

$\therefore$ Roots are $2, -5, -3$.

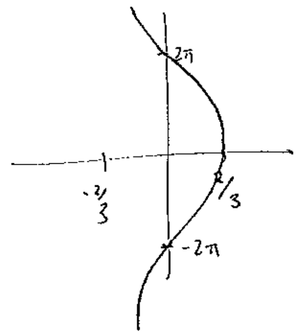
b) $y = 4\cos^{-1}\left(\frac{3x}{2}\right)$

$$\frac{y}{4} = \cos^{-1}\left(\frac{3x}{2}\right)$$ 4

$$\cos\left(\frac{y}{4}\right) = \frac{3x}{2}.$$

$$2\cos\left(\frac{y}{4}\right) = 3x.$$

$$\frac{2}{3}\cos\left(\frac{y}{4}\right) = x.$$



Domain $x: -\frac{2}{3} \leq x \leq \frac{2}{3}$

Range: $0 \leq y \leq 4\pi$

FOR IT TO BE STILL DEFINED AS A FUNCTION (i.e. passes straight line test).

$-4\pi \leq y \leq 0$ also acceptable.

c) 10 Rainy Days

and 20 non rainy days

$P(\text{10 rainy days}) = {}^{30}C_{10}\,(0.3)^{10}\,(0.7)^{20}$

≈ 0.1416

d) $P(2p, p^2)$

$Q(2q, q^2)$ $\quad x^2 = 4y$

$M = \dfrac{2p+2q}{2}, \dfrac{p^2+q^2}{2}$

i) $(p-q)^2 + (p+q)^2 = 2p^2+2q^2$

$p^2+q^2-2pq + p^2+q^2+2pq$

$= 2p^2+2q^2 = RHS$

ii) $x = \dfrac{2p+2q}{2}$, $y = \dfrac{p^2+q^2}{2}$

$= p+q$ $\quad = \dfrac{p^2+q^2}{2}$.

Simple substitution from i $\quad (p-q)^2 = 2(p^2+q^2) - (p+q)^2$

$p-q = 4$

$2(p^2+q^2) = 4y_m$

$p+q = x_m$

$\Rightarrow 4^2 = 4y_m - x_m^2$.

ie. $x_m^2 = 4y_m - 16$

$x^2 = 4y - 16$ for locus of M

iii)

$4y = x^2 + 16$

$y = \frac{1}{4}x^2 + 4$

$h = 0, k = 4$.

Focus is $\left(h, k + \frac{1}{4a}\right)$

$= \left(0, 4 + \frac{1}{4(\frac{1}{4})}\right)$

$= (0, 4+1)$

$= (0, 5)$

e) i) $\angle RPQ = \alpha$ (Alt. segment theorem)

$\therefore \angle SQP = \alpha$ (Isosceles ΔSPQ)

$\therefore \angle PSQ = 180° - 2\alpha$ ($\angle$ sum of ΔSPQ)

$\therefore \angle RSQ = 180 - (180 - 2\alpha)$ ($\angle$ sum of str. line)

$\therefore \angle RSQ = 2\alpha$.

ii) $RT = QT$ (2 tangents drawn from an external point to a circle are equal)

$\therefore \Delta RQT$ is isosceles.

$\therefore \angle TQR = \angle\alpha$.

$\therefore \angle RTQ = 180° - 2\alpha$ ($\angle$ sum of ΔRQT)

$\therefore$ SQTR is a cyclic quadrilateral. (Opposite angles are supplementary)

($\angle RTQ + \angle RSQ = 180° - 2\alpha + 2\alpha = 180° \Rightarrow$ Supplementary)

8.

a) i) $\int_0^{\pi/2} \sin^2 x\,dx = \frac{\pi}{4}$.

$\cos(2x) = 1-2\sin^2(x)$

$2\sin^2 x = 1-\cos(2x)$

$\sin^2 x = \frac{1-\cos(2x)}{2}$

$= \frac{1}{2}\int_0^{\pi/2} 1-\cos(2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\left[\frac{\pi}{2} - 0\right] - [0-0]\right]$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$.

ii) $y = \sin(x)+1$

Volume of revolution

$= \pi\int_0^{\pi/2} (\sin x + 1)^2\,dx$.

$= \pi\int_0^{\pi/2} \sin^2 x + 2\sin x + 1\,dx$

$= \pi\left[\frac{\pi}{4}\right] + \pi\left[-2\cos x + x\right]_0^{\pi/2}$

$= \frac{\pi^2}{4} + \pi\left[\frac{\pi}{2} + 2\right]$

$= \frac{\pi^2}{4} + \frac{\pi^2}{2} + 2\pi$

$= \frac{3\pi^2}{4} + 2\pi$.

b. $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots {}^nC_n x^n$

let $x = -1$

$1 + {}^nC_1(-1) + {}^nC_2(1) + {}^nC_3(-1)\ldots + {}^nC_n(-1)^n$

$= 0$

ii) ${}^nC_1 + 2\,{}^nC_2 + \ldots + n\,{}^nC_n = n2^{n-1}$

$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 \ldots$

$n(1+x)^{n-1} = 1 + {}^nC_1 + 2\,{}^nC_2 x + \ldots + n\,{}^nC_n x^{n-1}$

let $x = 1$

$n(2^{n-1}) = {}^nC_1 + 2\,{}^nC_2 + \ldots + n\,{}^nC_n$

c) i $P(\text{success}) = \frac{95}{100}$

$P(\text{Fail}) = \frac{5}{100}$

$P = {}^{40}C_{36}\left(\frac{95}{100}\right)^{36}\left(\frac{5}{100}\right)^4$

$= 0.0901$ (4dp.)

ii) $P(2\text{ max}) = P(0\text{ miss}) + P(1\text{ miss}) + P(2\text{ miss})$

$= {}^{40}C_0\left(\frac{95}{100}\right)^{40} + {}^{40}C_1\left(\frac{5}{100}\right)^1\left(\frac{95}{100}\right)^{39} + {}^{40}C_2\left(\frac{5}{100}\right)^2\left(\frac{95}{100}\right.$

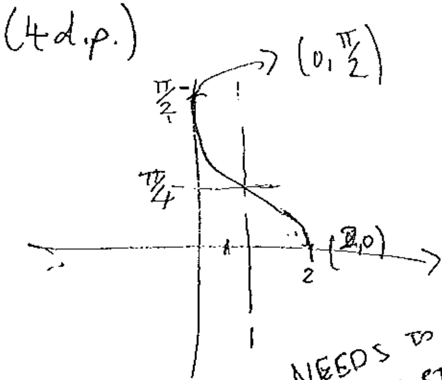
$= 0.6767$ (4d.p.)

d) $y = \frac{1}{2}\cos^{-1}(x-1)$

$2y = \cos^{-1}(x-1)$

$\cos(2y) = (x-1)$

$\cos(2y) + 1 = x$.



NEEDS TO SATISFY STRAIGHT LINE TEST!

Domain $-1 \le x-1 \le 1$

$0 \le x \le 2$

Range $0 \le 2y \le \pi$

$= 0 \le y \le \frac{\pi}{2}$

$$= \pi \int_0^{\pi/2} (\cos(2y)+1)^2 \, dy \qquad x = \cos(2y)+1$$

$$= \pi \int_0^{\pi/2} \cos^2(2y) + 1 + 2\cos(2y) \, dy$$

$$= \pi \int_0^{\pi/2} \tfrac{1}{2}(1+\cos(4y)) + 1 + 2\cos 2y \, dy$$

$$= \frac{\pi}{2} \int_0^{\pi/2} 1 + \cos 4y + 2 + 4\cos 2y \, dy$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \cos 4y + 4\cos 2y + 3 \, dy$$

$$= \frac{\pi}{2} \left[\tfrac{1}{4}\sin 4y + 2\sin 2y + 3y \right]_0^{\pi/2}$$

$$\frac{\pi}{2} \left[\tfrac{1}{4}\sin 2\pi + 2\sin \pi + 3\left(\tfrac{\pi}{2}\right) \right]$$

$$\frac{\pi}{2} \left(\frac{3\pi}{2} \right] \Rightarrow \frac{3\pi^2}{4} \, u^3$$