Student Number: ____________________

Year 12 Mathematics
Extension 1
Task 3
May 2017

ɜt Catherine's School
Waverley

ime allowed:	90 minutes plus 5 minutes reading time
otal marks:	52 marks
Veighting:	25%

here are 10 questions including 7 multiple choice questions.

ꟼSTRUCTIONS

- In Section I, answer the 7 multiple choice questions on the attached multiple choice answer sheet.
- In Section II, start Questions 8, 9 and 10 in three separate answer booklets. You may request more booklets if needed.
- Show all necessary working for Questions 8, 9 and 10.
- Marks may be deducted for careless or badly arranged work.
- A reference sheet is provided.
- Write using blue or black pen. Black pen is preferred.

Multiple Choice		/ 7 marks
Question 8		/15 marks
Question 9		/15 marks
Question 10		/15 marks
Total		**/52 marks**

SECTION I

7 marks

Attempt Questions 1 – 7

Use the multiple-choice answer sheet for Questions 1 – 7.

1. A particle is moving in a straight line. Its velocity v m/s at a position x metres from an origin O is given by $v^2 = 4(27 - 3x^4)$. What is the acceleration of the particle when it is 1 metre to the left of the origin?

(A) -24

(B) -48

(C) 24

(D) 48

2. A particle is moving in a straight line with its velocity as a function of x given by $\dot{x} = e^{-x}$. It is initially at the origin. The displacement x is given by

(A) $x = \log_e(t-1)$

(B) $x = \log_e(t+1)$

(C) $x = e^{t+1}$

(D) $x = -e^{-t} + 1$

3. The velocity, v ms^{-1}, of a particle moving in simple harmonic motion along the x-axis given by $v^2 = 8 - 2x - x^2$, where x is in metres. What is the amplitude of motion?

(A) 1

(B) 4

(C) -3

(D) 3

4. $\int \sin^2\left(\frac{x}{2}\right)\,dx$

(A) $\frac{x}{2}-\frac{1}{2}\sin x+c$

(B) $\frac{x}{2}-\frac{1}{2}\sin\frac{x}{2}+c$

(C) $\cos^3\left(\frac{x}{2}\right)+c$

(D) $x+\frac{1}{2}\sin x+c$

5. Which integral is obtained when the substitution $u=\cos^{-1}x$ is applied to

$$\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$$

(A) $\int e^u\,du$

(B) $-\int e^u\,du$

(C) $\int \frac{e^u}{\sqrt{1-\cos^2 u}}\,du$

(D) $\int \frac{e^{\sin u}}{\sqrt{1-\cos^2 u}}\,du$

6. $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x\,dx$

(A) $\frac{1}{3}$

(B) 1

(C) 3

(D) 0

7. A spherical bubble is expanding so that its volume is increasing at the constant rate of $10\ \text{cm}^3$ per second. What is the rate of increase of the radius when the surface area is $400\ \text{cm}^2$?

(A) $4\ \text{cm/s}$

(B) $40\ \text{cm/s}$

(C) $\frac{1}{10}\ \text{cm/s}$

(D) $\frac{1}{40}\ \text{cm/s}$

End of Section I

SECTION II 45 marks

Attempt Questions 8 – 10

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 8 – 10, your response should include relevant mathematical reasoning and/or calculations.

Question 8 Use a SEPARATE writing booklet **15 marks**

(a) Use the substitution $u = 1 + x$ to find **3**

$$\int_1^2 \frac{1-x}{(1+x)^3}\,dx$$

(b) A salad, which is initially at a temperature of 23°C, is placed in a refrigerator that has a constant temperature of 4°C. The cooling rate of the salad is proportional to the difference between the temperature of the refrigerator and the temperature, T, of the salad. That is, T satisfies the equation

$$\frac{dT}{dt} = -k(T-4)$$

where t is the number of minutes after the salad is placed in the refrigerator.

(i) Show that $T = 4 + Be^{-kt}$ satisfies the equation. **1**

(ii) The temperature of the salad is 12°C after 8 minutes. Find the values of B and k. **2**

(iii) How long will it take for the temperature of the salad to cool to 9°C ? Give your answer correct to the nearest minute. **2**

(iv) Draw a graph of T against t, showing key points and features. **2**

Question 8 continues

Question 8 (continued)

(c) The velocity of a particle moving in a straight line is given by $v = (x+1)^2$ where x is its displacement from the origin after t seconds. Initially the particle is at the origin.

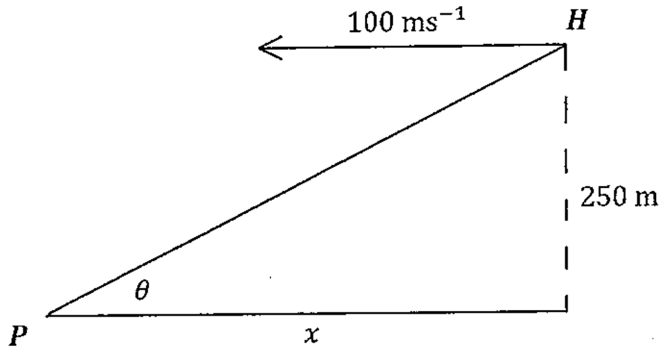
(i) Find an expression for a in terms of x. **2**

(ii) Find an expression for x in terms of t. **3**

End of Question 8

Question 9 Use a SEPARATE writing booklet **15 marks**

(a) A person P on horizontal ground is looking up at a helicopter H which is approaching at a speed of $100\ \text{ms}^{-1}$ at a constant altitude of 250 metres above the ground. When the horizontal distance of the helicopter from the person is x metres, the angle of elevation of the helicopter is θ radians.



(i) Show that $\theta = \tan^{-1}\frac{250}{x}$ **1**

(ii) Show that $\frac{d\theta}{dt} = \frac{25000}{x^2+62500}$ **3**

(iii) Find the rate at which θ is changing when $\theta = \frac{\pi}{4}$, giving the answer in degrees per second correct to the nearest degree. **2**

(b) A particle moves in a straight line. Its displacement, x metres, after t seconds is given by

$$x = \sin 2t - \sqrt{3}\cos 2t$$

(i) Express the displacement of the particle in the form $x = A\sin(2t - \alpha)$ where α is in radians. **2**

(ii) Show that the particle is moving in simple harmonic motion. **2**

(iii) What is the period of the motion? **1**

(iv) Find the maximum speed of the motion. **1**

Question 9 continues

Question 9 (continued)

(c) A particle moving in a straight line with SHM has a period of π seconds and oscillates between $x = -2$ and $x = 4$. The particle is initially 1 metre to the right of the origin. **3**

Write an expression for the displacement x in terms of t as
$x = a\cos(nt + \alpha) + c$.

End of Question 9

Question 10 Use a SEPARATE writing booklet **15 marks**

(a) Use the substitution $u^2 = x, u > 0$, evaluate **3**

$$\int \frac{1}{4(x+\sqrt{x})}\,dx$$

(b) Using the substitution $x = 5\sin\theta$ to find **4**

$$\int \sqrt{25 - x^2}\,dx$$

(c) An arrow is fired from the origin O with initial velocity $80\ ms^{-1}$ with variable angle of projection θ to hit a target $40\ m$ high and $200\ m$ away from the point of projection. The equations of motion are given by

$$x = 80t\cos\theta$$

$$y = 80t\sin\theta - 5t^2$$

Do NOT prove these equations of motion.

(i) Show that the Cartesian equation of motion is given by **2**

$$y = x\tan\theta - \frac{x^2}{1280}(1 + \tan^2\theta)$$

(ii) Show that the range of the motion is given by $640\sin 2\theta$. **2**

(iii) Find the two possible angles of projection, θ, so that the arrow hits the target. **2**

Give your answers correct to the nearest minute.

(iv) If the arrow just misses the target, find the closest distance the arrow lands from the target. **2**

Give your answer correct to 2 decimal places.

End of paper

Student Number: SOLUTIONS

St Catherine's School
Waverley

Year 12 Mathematics
Extension 1
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Multiple Choice Answer Sheet

Colour in the correct oval completely

1. A ○ B ○ C ● D ○

2. A ○ B ● C ○ D ○

3. A ○ B ○ C ○ D ●

4. A ● B ○ C ○ D ○

5. A ○ B ● C ○ D ○

6. A ● B ○ C ○ D ○

7. A ○ B ○ C ○ D ●

St Catherine's School
Waverley

Student number: SOLUTIONS

Course name: Mathematics Extension 1

Question: Multiple Choice 1 – 7

(1) $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$

$= \frac{d}{dx}\left[2(27-3x^4)\right]$

$= 2(-12x^3)$

$= -24x^3$

when $x=-1$, $\ddot{x} = -24(-1)^3$

$= 24$

(2) $\frac{dx}{dt} = e^{-x}$

$\frac{dt}{dx} = e^{x}$

$dt = e^{x}\,dx$

$t = \int e^{x}\,dx$

$t = e^{x} + c$

when $t=0$, $x=0 \Rightarrow 0 = e^0 + c$

$c = -1$

$t = e^{x} - 1$

$e^{x} = t+1$

$x = \ln(t+1)$

(3) $v^2 = 8 - 2x - x^2$

$= -(x^2 + 2x - 8)$

$= -(x^2 + 2x + 1 - 9)$

$= -\left[(x+1)^2 - 3^2\right]$

$= 1^2\left[3^2 - (x+1)^2\right]$

comparing $v^2 = n^2[a^2-(x-c)^2]$ to $v^2 = 1^2[3^2-(x+1)^2]$

$\therefore$ amplitude $a = 3$

OR

extreme points of motion when $v = 0$

then $0 = 8-2x-x^2$

$0 = x^2+2x-8$

$0 = (x+4)(x-2)$

$x = 2$ or $x = -4$

amplitude $a = \dfrac{2+|-4|}{2}$

$= \dfrac{6}{2}$

$\therefore$ $a = 3$

(4) $\cos 2\theta = 1-2\sin^2\theta$

$2\sin^2\theta = 1-\cos 2\theta$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin^2(\frac{x}{2}) = \frac{1}{2}(1-\cos x)$ when $\theta = \frac{x}{2}$

then $\int \sin^2(\frac{x}{2})\,dx = \frac{1}{2}\int 1-\cos x \, dx$

$= \frac{1}{2}(x-\sin x) + c$

$= \frac{x}{2} - \frac{1}{2}\sin x + c$

(5) let $u = \cos^{-1}x$

$x = \cos u$

$\frac{dx}{du} = -\sin u$

$dx = -\sin u \, du$

$\int \dfrac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx = \int \dfrac{e^u}{\sqrt{1-\cos^2 u}} \cdot -\sin u \, du$

$= -\int \dfrac{e^u}{\sqrt{\sin^2 u}} \cdot \sin u \, du$

$$= -\int \frac{e^u}{\cancel{\sin u}} \cdot \cancel{\sin u \, du}$$

$$= -\int e^u \, du$$

(6) $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx = \frac{1}{3}\left[\sin^3 x\right]_0^{\frac{\pi}{2}}$

$$= \frac{1}{3}\left[\left(\sin\frac{\pi}{2}\right)^3 - (\sin 0)^3\right]$$

$$= \frac{1}{3}\left(1^3 - 0^3\right)$$

$$= \frac{1}{3}$$

(7) given $\frac{dV}{dt} = 10$

find $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

now Surface Area $= \frac{dV}{dr}$

$= 400$

$\therefore \frac{dr}{dt} = \frac{1}{400} \times 10$

$= \frac{1}{40}$ cm/s

St Catherine's School

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Student number: SOLUTIONS

Course name: Mathematics Extension 1

Question: 8

(a) let $u = 1+x$ when $x=1$, $u = 1+1 = 2$
$\frac{du}{dx} = 1$ when $x = 2$, $u = 1+2 = 3$ $\frac{1}{2}$
$du = dx$

$$\int_1^2 \frac{1-x}{(1+x)^3}\,dx = \int_2^3 \frac{1-(u-1)}{u^3}\,du$$

$$= \int_2^3 \frac{1-u+1}{u^3}\,du$$

$$= \int_2^3 \frac{2-u}{u^3}\,du \quad 1$$

$$= \int_2^3 \frac{2}{u^3} - \frac{u}{u^3}\,du$$

$$= \int_2^3 2u^{-3} - u^{-2}\,du$$

$$= \left[-u^{-2} + u^{-1}\right]_2^3$$

$$= \left[-\frac{1}{u^2} + \frac{1}{u}\right]_2^3 \quad 1$$

$$= -\frac{1}{3^2} + \frac{1}{3} - \left(-\frac{1}{2^2} + \frac{1}{2}\right)$$

$$= -\frac{1}{36} \quad \frac{1}{2}$$

(b) (i) $T = 4 + Be^{-kt} \Rightarrow Be^{-kt} = T - 4$

$\frac{dT}{dt} = -kBe^{-kt}$

$\therefore \frac{dT}{dt} = -k(T-4)$ as required (1)

(ii) when $t=0$, $T=23$

$23 = 4 + Be^{-k \times 0}$

$23 = 4 + Be^{0}$

$23 = 4 + B$

$\therefore B = 19$ (1)

then $T = 4 + 19e^{-kt}$

when $t=8$, $T=12$

$12 = 4 + 19e^{-k \times 8}$

$8 = 19e^{-8k}$

$\frac{8}{19} = e^{-8k}$

$\frac{8}{19} = \frac{1}{e^{8k}}$

$e^{8k} = \frac{19}{8}$

$8k = \ln\frac{19}{8}$

$\therefore k = \frac{1}{8}\ln\frac{19}{8}$ (1)

(iii) when $T=9$, $t=?$

$9 = 4 + 19e^{(-\frac{1}{8}\ln\frac{19}{8})t}$

$5 = 19e^{(\frac{1}{8}\ln\frac{8}{19})t}$

$\frac{5}{19} = e^{(\frac{1}{8}\ln\frac{8}{19})t}$

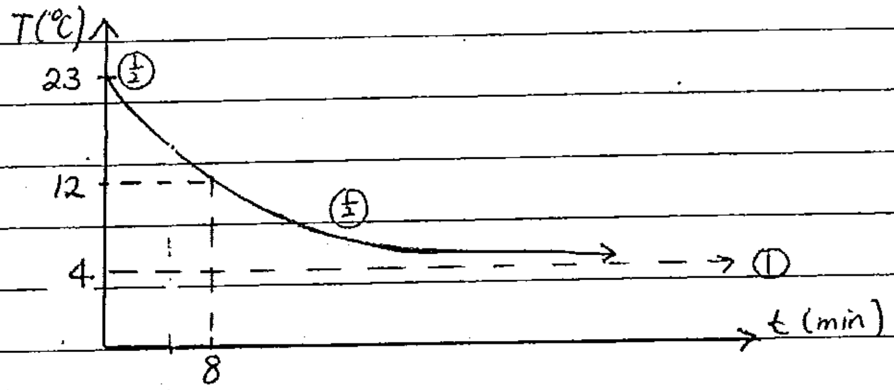
$(\frac{1}{8}\ln\frac{8}{19})t = \ln\frac{5}{19}$ (1)

$t = \ln\frac{5}{19} \div \frac{1}{8}\ln\frac{8}{19}$

$= 12°20'48''$

$\therefore t \doteqdot 12$ minutes (nearest minute) (1)

(iv)



(c) (i) $a = v\dfrac{dv}{dx}$

$= (x+1)^2 \dfrac{d}{dx}(x+1)^2$ ①

$= (x+1)^2 \left(2(x+1)\right)$

$\therefore\ a = 2(x+1)^3$ ①

(ii) $v = \dfrac{dx}{dt}$

$\dfrac{dx}{dt} = (x+1)^2$

$\dfrac{dt}{dx} = (x+1)^{-2}$

$t = \int (x+1)^{-2}\,dx$ ①

$= -(x+1)^{-1} + c$

$t = -\dfrac{1}{x+1} + c$

when $t=0,\ x=0$

$0 = -\dfrac{1}{0+1} + c$

$c = 1$

then $t = -\dfrac{1}{x+1} + 1$ ①

$\dfrac{1}{x+1} = 1-t$

$x+1 = \dfrac{1}{1-t}$

$\therefore\ x = \dfrac{1}{1-t} - 1$ ①

OR

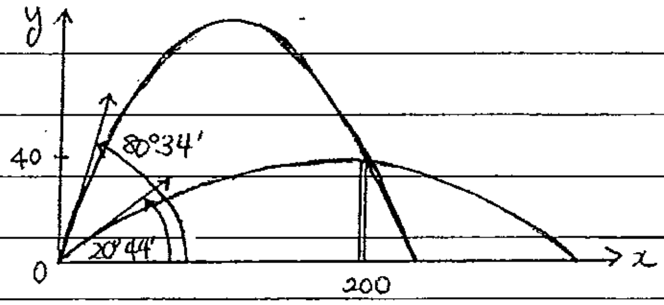
$x = \dfrac{t}{1-t}$

(iv) the arrow that lands closest to the target is when angle of projection $\theta = 80°34'$ (½)

then range when $\theta = 80°34'$:

$$x = 640\sin 2(80°34')$$
$$= 640\sin(161°8')$$
$$= 206.954\ldots$$ (1)

then distance from target $= 206.954\ldots - 200$
$= 6.954\ldots$
$= 6.95$ m (2 decimal places) (½)



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Student number: SOLUTIONS

Course name: Mathematics Extension 1

Question: 9

(a) (i) $\tan\theta = \frac{250}{x}$

$\therefore \theta = \tan^{-1}\frac{250}{x}$ (1)

(ii) $\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{250}{x}\right)^2} \times \left(-250x^{-2}\right)$ (1)

$= \frac{1}{\frac{x^2+250^2}{x^2}} \times -\frac{250}{x^2}$

$= \frac{\cancel{x^2}}{x^2+250^2} \times \frac{-250}{\cancel{x^2}}$

$= -\frac{250}{x^2+62500}$ (1)

given $\frac{dx}{dt} = -100$

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$= -\frac{250}{x^2+62500} \times -100$ (1)

$\therefore \frac{d\theta}{dt} = \frac{25000}{x^2+62500}$ as required

(iii) when $\theta = \frac{\pi}{4}$, $\frac{250}{x} = \tan\theta \Rightarrow \frac{250}{x} = \tan\frac{\pi}{4} \Rightarrow \frac{250}{x} = 1$

$x = 250$ (1)

$\frac{d\theta}{dt} = \frac{25000}{250^2+62500} = \frac{1}{5}$ (1)

(b) (i) $\sin 2t - \sqrt{3}\cos 2t = A\sin(2t-\alpha)$

$= A\sin 2t\cos\alpha - A\cos 2t\sin\alpha$

$= (A\cos\alpha)\sin 2t - (A\sin\alpha)\cos 2t$

comparing coefficient of LHS and RHS

$A\cos\alpha = 1$ ① and $A\sin\alpha = \sqrt{3}$ ②

then $①^2 + ②^2$: $A^2\cos^2\alpha + A^2\sin^2\alpha = 1^2 + \sqrt{3}^2$

$A^2(\cos^2\alpha + \sin^2\alpha) = 4$

$A^2 = 4$

$A = 2$ ①

and ② ÷ ① : $\dfrac{A\sin\alpha}{A\cos\alpha} = \dfrac{\sqrt{3}}{1}$

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$ ①

$\therefore x = 2\sin(2t - \frac{\pi}{3})$

(ii) $\dot{x} = 4\cos(2t - \frac{\pi}{3})$

$\ddot{x} = -8\sin(2t - \frac{\pi}{3})$ ①

$= -4[2\sin(2t - \frac{\pi}{3})]$

$= -4x$

$\therefore \ddot{x} = -2^2 x$ ①

$\therefore$ the particle moves in SHM with $n = 2$

(iii) $T = \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{2}$

$\therefore T = \pi$ seconds ①

(iv) $\dot{x} = 4\cos\left(2t - \frac{\pi}{3}\right)$

max. value of $\cos\left(2t - \frac{\pi}{3}\right) = 1$

$\therefore$ max. $|\dot{x}| = 4 \times 1$

$= 4$ m/s ①

(c) $T = \frac{2\pi}{n}$

$\pi = \frac{2\pi}{n}$

$\therefore n = 2$ ($\frac{1}{2}$)

$2a = |-2| + 4$

$2a = 6$

$\therefore a = 3$ ($\frac{1}{2}$)

centre of motion $= \frac{-2+4}{2}$

$\therefore c = 1$ ($\frac{1}{2}$)

then $x = 3\cos(2t + \alpha) + 1$

when $t = 0, x = 1$:

$1 = 3\cos(2 \times 0 + \alpha) + 1$

$0 = 3.\cos\alpha$

$0 = \cos\alpha$

$\therefore \alpha = \frac{\pi}{2}$ ①

$\therefore x = 3\cos\left(2t + \frac{\pi}{2}\right) + 1$ ($\frac{1}{2}$)

OR

$x = 3\cos 2\left(t + \frac{\pi}{4}\right) + 1$

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Student number: SOLUTIONS
Course name: Mathematics Extension 1
Question: 10

(a) let $u^2 = x$

$u = \sqrt{x}$ since $u > 0$

and $\frac{dx}{du} = 2u$

$dx = 2u\, du$ (1/2)

$$\int \frac{1}{4(x+\sqrt{x})}\,dx = \int \frac{\cancel{2}u}{{}^{2}\cancel{4}(u^2+u)}\,du$$

$$= \frac{1}{2}\int \frac{\cancel{u}}{\cancel{u}(u+1)}\,du$$

$$= \frac{1}{2}\int \frac{1}{u+1}\,du \quad (1\tfrac{1}{2})$$

$$= \frac{1}{2}\ln(u+1) + c$$

$$= \frac{1}{2}\ln(\sqrt{x}+1) + c \quad (1)$$

(b) let $x = 5\sin\theta$

$\frac{dx}{d\theta} = 5\cos\theta$

$dx = 5\cos\theta\, d\theta$

$$\int \sqrt{25 - x^2}\,dx = \int \sqrt{25 - 25\sin^2\theta} \cdot 5\cos\theta\, d\theta \quad (1)$$

$$= \int \sqrt{25(1-\sin^2\theta)} \cdot 5\cos\theta\, d\theta$$

$$= \int 5\sqrt{\cos^2\theta} \cdot 5\cos\theta\, d\theta$$

$$= 25\int \cos\theta \cdot \cos\theta\, d\theta$$

$$= 25\int \cos^2\theta\, d\theta$$

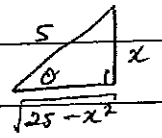
$$= \frac{25}{2}\int \cos 2\theta + 1\, d\theta \quad \text{since } \cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{25}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c \quad (1)$$

p.10

now $x = 5\sin\theta$

$\frac{x}{5} = \sin\theta \implies \theta = \sin^{-1}\frac{x}{5}$



$\cos\theta = \frac{\sqrt{25-x^2}}{5}$

then $\frac{25}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c$

$= \frac{25}{2}\left(\frac{1}{2}\cdot 2\sin\theta\cos\theta + \theta\right) + c$ (1)

$= \frac{25}{2}\left(\sin\theta\cos\theta + \theta\right) + c$

$= \frac{25}{2}\left(\frac{x}{5} \times \frac{\sqrt{25-x^2}}{5} + \sin^{-1}\frac{x}{5}\right) + c$

$= \frac{25}{2}\left(\frac{x\sqrt{25-x^2}}{25} + \sin^{-1}\frac{x}{5}\right) + c$

$= \frac{x\sqrt{25-x^2}}{2} + \frac{25}{2}\sin^{-1}\frac{x}{5} + c$ (1)

$\therefore \int \sqrt{25-x^2}\,dx = \frac{x\sqrt{25-x^2}}{2} + \frac{25}{2}\sin^{-1}\frac{x}{5} + c$

(c) (i) $x = 80t\cos\theta$

$t = \frac{x}{80\cos\theta}$ (1)

substitute (1) into $y = 80t\sin\theta - 5t^2$

$y = 80\left(\frac{x}{80\cos\theta}\right)\sin\theta - 5\left(\frac{x}{80\cos\theta}\right)^2$ (1)

$= x\tan\theta - \frac{5x^2}{6400\cos^2\theta}$

$= x\tan\theta - \frac{x^2}{1280}\sec^2\theta$ (1)

$\therefore y = x\tan\theta - \frac{x^2}{1280}(1+\tan^2\theta)$ as required

(ii) for range $y = 0$

$$0 = 80t\sin\theta - 5t^2$$

$$0 = 5t(16\sin\theta - t)$$

$$t = 0 \quad \text{or} \quad t = 16\sin\theta \qquad 1$$

(initial) (range)

when $t = 16\sin\theta$, $x = 80(16\sin\theta)\cos\theta$

$$= 1280\sin\theta\cos\theta$$

$$= 640 \times 2\sin\theta\cos\theta \qquad 1$$

$\therefore$ $x = 640\sin 2\theta$ is the range

(iii) when $x = 200$ and $y = 40$

$$40 = 200\tan\theta - \frac{200^2}{1280}(1+\tan^2\theta)$$

$$40 = 200\tan\theta - \frac{125}{4}(1+\tan^2\theta)$$

$$160 = 800\tan\theta - 125 - 125\tan^2\theta$$

$$32 = 160\tan\theta - 25 - 25\tan^2\theta$$

$$0 = 25\tan^2\theta - 160\tan\theta + 57 \qquad 1$$

$$\tan\theta = \frac{160 \pm \sqrt{(-160)^2 - 4(25)(57)}}{2(25)}$$

$$= \frac{160 \pm \sqrt{19900}}{50}$$

$$= \frac{16\cancel{0} \pm 1\cancel{0}\sqrt{199}}{5\cancel{0}}$$

$$\tan\theta = \frac{16 + \sqrt{199}}{5} \quad \text{or} \quad \tan\theta = \frac{16 - \sqrt{199}}{5}$$

$$\theta = \tan^{-1}\left(\frac{16+\sqrt{199}}{5}\right) \qquad \theta = \tan^{-1}\left(\frac{16-\sqrt{199}}{5}\right)$$

$\therefore$ $\theta = 80°34'$ or $\theta = 20°44'$ 1 p.12