Student's Number:

Year 12 Mathematics

HSC Task 3

May 2017

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- Marks may be deducted for careless or badly arranged work
- Task Weighting – 25%
- Total Marks – 50

SECTION I **7 marks**

- Attempt Questions 1 – 7 on Multiple Choice Answer Sheet.

SECTION II **60 marks**

- Attempt Questions 8 – 11
- Answer each question in a separate booklet
- Show all necessary working

Question 1 – 7		/7
Question 8		/15
Question 9		/15
Question 10		/15
Question 11		/15
TOTAL		**/67**

Section I

Use the multiple-choice answer sheet for questions 1 – 7.

1 What is the minimum value of the function

$y = 2\cos x + 3$

A 1

B 5

C -1

D -7

2 What is the value of $\int_2^6 \frac{1}{x+2}\,dx$

A $\ln 2$

B $\ln 4$

C $\frac{-1}{8}$

D $\ln 32$

3 What are the solutions to

$e^{2x} - 5e^x + 6 = 0$

A $x = \frac{3}{e}$ and $x = \frac{2}{e}$

B $x = 2$ and $x = 3$

C $x = \ln 2$ and $x = \ln 3$

D $x = e^2$ and $x = e^3$

4 For the graph of $y = 1 + 3\cos(2x - \frac{\pi}{4})$,

A The amplitude is 4 and the period is π

B The amplitude is 1 and the period is 2π

C The amplitude is 3 and the period is π

D The amplitude is 3 and the period is 2π

5 The gradient of a function $y = f(x)$ is given by $y' = (e^x - 1)(x - 2)$

Which of the following is true for $y = f(x)$?

A The only stationary point is $x = 2$

B The stationary points are $x = 1$ and $x = 2$

C The stationary points are $x = \ln e$ and $x = 2$

D The stationary points are $x = 0$ and $x = 2$

6 The solution to the equation $\ln 2 + \ln x - \ln(x + 2) = 0$

A $x = 2$

B $x = 0$

C $x = e^{-\ln 2}$

D $x = \ln 2$

7 $$\int \cos x^0 \, dx$$

A $\sin x^0 + C$

B $\frac{\pi}{180} \sin x^0 + C$

C $\frac{180}{\pi} \sin x^0 + C$

D $-\sin x^0 + C$

Please turn over for Section II

Section II

60 marks

Attempt questions 8 – 11

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 8 – 11, your responses should include relevant mathematical reasoning and/or calculations.

Question 8: 15 marks

Marks

a) Differentiate the following:

(i) $y = \ln(x^2 + 1)$ 1

(ii) $y = \sin(\ln x)$ 2

(iii) $y = e^{2x} \cos x$ 2

(iii) $y = sin^2 x$ 2

(iv) $y = \frac{ln\, x}{x}$ 2

b) (i) Show that $\int_5^7 \frac{x-1}{x^2-2x}\, dx = \frac{1}{2}\ln\frac{7}{3}$ 3

(ii) Find the exact value of $\int_0^{\ln 7} e^{-2x}\, dx$ 3

Please turn over for Question 9

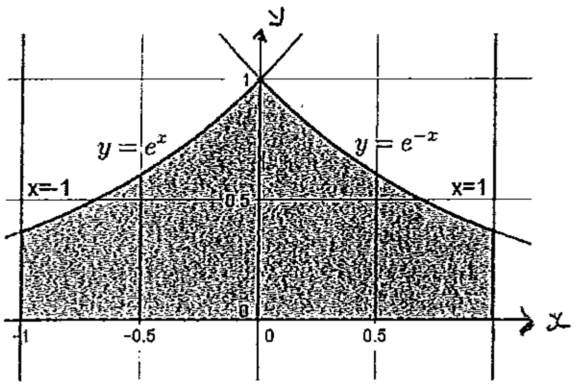
Question 9: 15 marks **Start a new booklet** **Marks**

a) (i) $y = x\ln x + x$

Show that $\frac{dy}{dx} = \ln x + 2$ 1

(ii) Hence integrate $\int_1^e \ln x\, dx$ 3

b) The graph below shows the region bounded by $y = e^x$, $y = e^{-x}$, the $x\ axis$, the lines $x = -1$ and $x = 1$



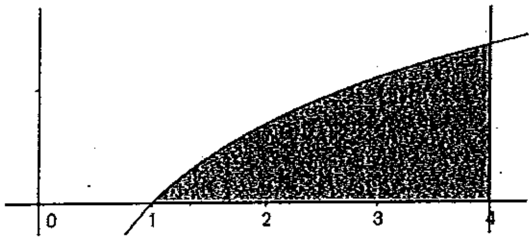
(i) Show that the shaded area is $2 - \frac{2}{e}$ units^2. 3

c) Estimate $\int_0^1 \sin(1 + x^2)dx$ by using Simpson's rule with three function values to 2 decimal places. 3

Question 9 continues on the next page

Question 9 continued

d) The sketch shows the graph of $y = ln\sqrt{x}$



(i) Show that $x = e^{2y}$ 1

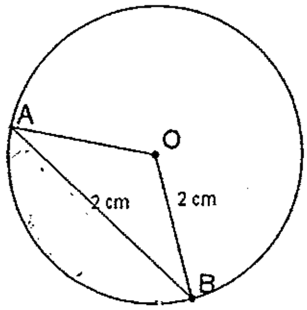
(ii) Show that the area bounded by $y = ln\sqrt{x}$, the x axis and $x = 4$ is given by $(4\ln 2 - \frac{3}{2})$ units^2. 4

Please turn over for Question 10

Question 10: **15 marks** **Start a new booklet** **Marks**

a) Evaluate $\lim_{x \to 0} \frac{\sin 5x}{2x}$ 2

b) The diagram shows a circle with centre O and radius 2 cm. Chord AB is also 2 cm in length.



(i) Show that the area of the sector AOB is $\frac{2\pi}{3}$ cm^2. 2

(ii) Hence or otherwise find the exact area of the shaded segment. 2

c) Sketch the graph of $y = 3\sin(x - \frac{\pi}{6})$, $0 \le x \le 2\pi$, showing clearly the coordinates of the end points, the x-intercepts and the maximum and minimum turning points. 3

Question 10 is continued on the next page

Question 10 continued

d) Consider the function $y = x^2e^x$,

(i) Show that $\frac{dy}{dx} = xe^x(2 + x)$ **1**

(ii) Find the coordinates of the stationary points in exact form. **3**

Using the sign of the gradient function or otherwise, determine the nature of the stationary points.

(iii) What is the limiting value of $y = x^2e^x$ when x tends to $-\infty$. **1**

(iv) Sketch the function $y = x^2e^x$ showing all the above features. **1**

Please turn over for Question 11

Question 11: **15 marks** **Start a new booklet**

a) Differentiate $y = ln\frac{(x-1)}{\sqrt{x}}$ (Hint: Use log laws to simplify first) 2

b) (i) Given $y = \frac{\sin x}{1+\cos x}$

show that $\frac{dy}{dx} = \frac{1}{1+\cos x}$ 2

(ii) Show that $\int_0^{\frac{\pi}{4}} \frac{1}{1+\cos x}\, dx = \sqrt{2} - 1$ 2

c) (i) Sketch the curve $y = \sqrt{3}\cos 2x$ and $y = \sin 2x$ on the same set of axes, $0 \leq x \leq \pi$. 1

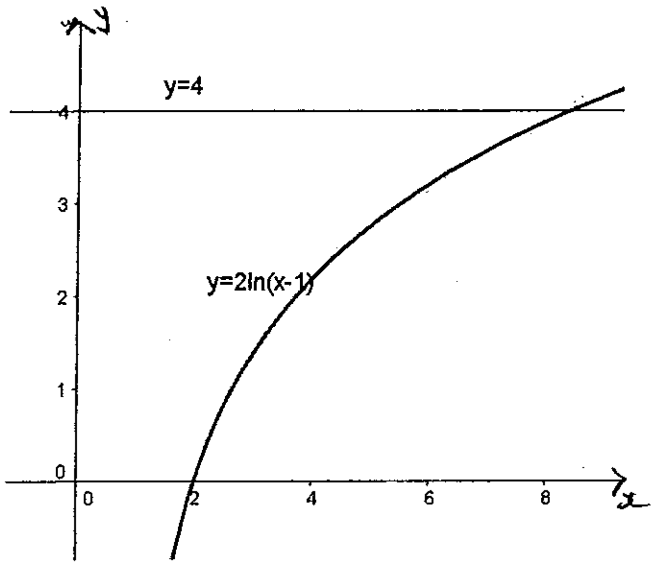
(ii) Show that the points of intersection of the two curves are $x = \frac{\pi}{6}$ and $x = \frac{2\pi}{3}$. 1

(ii) Find the area bounded by the two curves. 3

Question 11 is continued on the next page

Question 11 continued...

d) The curve $y = 2\ln(x-1)$, $0 \le y \le 4$ is rotated about the y axis.



Find the volume generated in exact form. 4

End of paper

Solution

St Catherine's School
Waverley

Student Number:

2017

HIGHER SCHOOL CERTIFICATE
MATHEMATICS

Multiple Choice Answer Sheet

Colour in the correct oval completely

1. A ● B ○ C ○ D ○
2. A ● B ○ C ○ D ○
3. A ○ B ○ C ● D ○
4. A ○ B ○ C ● D ○
5. A ○ B ○ C ○ D ●
6. A ● B ○ C ○ D ○
7. A ○ B ○ C ● D ○

Task 3 Mathematics 2017.

Qn	Solutions	Marks	Comments: Criteria
1.	$y = 2\cos x + 3$ minimum value of $\cos x = -1$. $\therefore$ y has a minimum value of $-2+3 = 1$. $\boxed{A}$		
2.	$\int_2^6 \frac{1}{x+2}\,dx = \left(\ln(x+2)\right)_2^6$ $= \ln 8 - \ln 4$ $= \ln\frac{8}{4} = \ln 2$. $\boxed{A}$		
3.	$e^{2x} - 5e^{x} + 6 = 0$ $(e^{x}-3)(e^{x}-2) = 0$ $e^{x} = 3$ $e^{x} = 2$ $x = \ln 3$ $x = \ln 2$ $\boxed{C}$		
4.	$y = 1 + 3\cos(2x - \frac{\pi}{4})$ Amplitude is 3 and Period is $\frac{2\pi}{2} = \pi$ $\boxed{C}$		
5.	Stat. pt. is when $y' = 0$ $e^{x} = 1$; $e^{x} =$ $\underline{x = 2}$. $x = \ln 1$. $\underline{x = 0}$. $\boxed{D}$		

Qn	Solutions	Marks	Comments: Criteria
6	$\ln \frac{2x}{x+2} = 0$ $\frac{2x}{x+2} = 1$ $2x = x+2$ $x = 2$ $\boxed{A}$		
7	$\int \cos x^\circ \, dx$ $180^\circ \quad \pi^c$ $x^\circ \quad \frac{\pi}{180} x^c$ $= \int \cos \frac{\pi x}{180} \, dx$ $= \frac{\sin \frac{\pi x}{180}}{\pi/180} + C = \frac{180}{\pi} \sin x^\circ + C$ $\boxed{C}$		

Qn	Solutions	Marks	Comments: Criteria
8.	$y = \ln(x^2+1)$ $y' = \dfrac{2x}{x^2+1}$	1	
ii)	$y = \sin(\ln x)$ $y' = \cos(\ln x)\cdot\dfrac{1}{x}$ $= \dfrac{1}{x}\cos(\ln x)$.	$\cos(\ln x)$ 2	1 M.
iii)	$y = e^{2x}\cdot\cos x$ $y' = 2e^{2x}\cos x - e^{2x}\sin x$ $= e^{2x}(2\cos x - \sin x)$	✓✓	
iv)	$y = \sin^2 x$ $y' = 2\sin x\cos x$	✓✓	
v)	$y = \dfrac{\ln x}{x}$ $y' = \dfrac{x\cdot\frac{1}{x} - \ln x}{x^2}$ $1\frac{1}{2}$ $= \dfrac{1-\ln x}{x^2}$. $\frac{1}{2}$		

Qn	Solutions	Marks	Comments: Criteria
b)	$\int_5^7 \frac{x-1}{x^2-2x}\,dx = \frac{1}{2}\int_5^7 \frac{2x-2}{x^2-2x}\,dx$		
	$= \frac{1}{2}\left(\ln(x^2-2x)\right)_5^7$	2	
	$= \frac{1}{2}(\ln 35 - \ln 15)$	$\frac{1}{2}$	
	$= \frac{1}{2}\ln\frac{35}{15}$		
	$= \frac{1}{2}\ln\frac{7}{3}$.	$\frac{1}{2}$	
c	$\int_0^{\ln 7} e^{-2x}\,dx = \left(\frac{e^{-2x}}{-2}\right)_0^{\ln 7}$	1	
	$= -\frac{1}{2}\left(e^{-2\ln 7} - e^0\right)$		
	$= -\frac{1}{2}\left(e^{\ln 7^{-2}} - 1\right)$	1	
	$= -\frac{1}{2}\left(\frac{1}{49} - 1\right)$		
	$= \frac{24}{49}$.	1	

Qn	Solutions	Marks	Comments: Criteria

Q. 9

a) $y = x\ln x + x$

$y' = x \cdot \frac{1}{x} + 1 . \ln x + 1$ 1

$= 2 + \ln x$

$\therefore \int_1^e (2 + \ln x)\, dx = (x\ln x + x)_1^e$

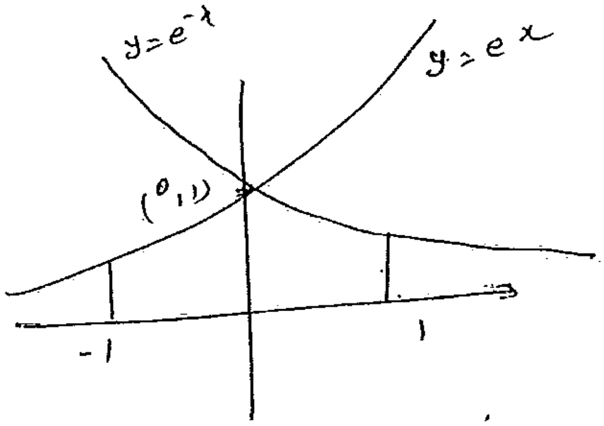
(ii) $\therefore \int_1^e \ln x\, dx = (x\ln x + x)_1^e - \int_1^e 2\, dx$ 1

$= ((e\ln e + e) - (1)) - (2e - 2)$

$= 2e - 1 - 2e + 2$

$= 1$

b)



The shaded area is

$\int_{-1}^{0} e^{x}\, dx + \int_0^1 e^{-x}\, dx$

$= (e^{x})_{-1}^{0} + \left(\frac{e^{-x}}{-1}\right)_0^1$

$= (e^{0} - e^{-1}) - (e^{-1} - e^{0})$

$= 1 - \frac{1}{e} - \frac{1}{e} + 1 = 2 - \frac{2}{e}$

$= 1.2642$

Qn	Solutions	Marks	Comments: Criteria

c)

$0 \quad 0.5 \quad 1$

$h = 0.5 \qquad \sin(1+0). = \sin 1$

$\sin(1+0.5^2) = \sin 1.25$

$\sin(1+1^2) = \sin 2$

$\therefore \int_0^1 \sin(1+x^2)\,dx$

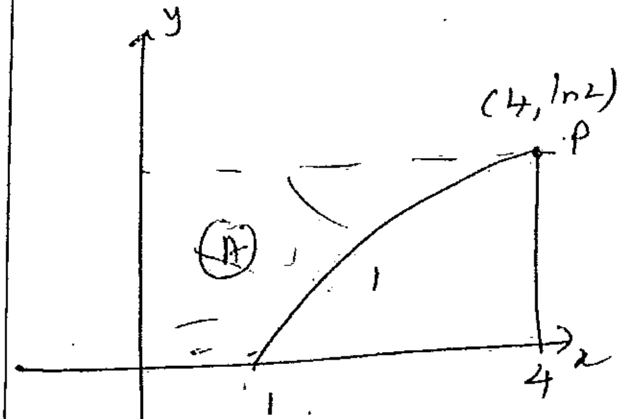
$\doteqdot \frac{0.5}{3}\left(\sin 1 + 4\sin 1.25 + \sin 2\right)$ 2M

$(2.d.p.)$ (M)

$= 0.92$

$(= 0.023 \text{ in degrees})$ 0 1/2

d)



$y = \ln\sqrt{x}$

$y = \frac{1}{2}\ln x$

$2y = \ln x$

$x = e^{2y}$ (1M)

Note: $P: (4, \ln\sqrt{4}) : (4, \ln 2)$

(Shaded area = Area of rectangle − (A)) (1M)

Area (A) $= \int_0^{\ln 2} x\,dy = \int_0^{\ln 2} e^{2y}\,dy$ (1M)

$= \frac{1}{2}\left(e^{2y}\right)_0^{\ln 2}$ (1M)

$= \frac{1}{2}\left(e^{2\ln 2} - e^0\right)$

$= \frac{1}{2}\left(e^{\ln 4} - 1\right) = \frac{3}{2}$ (1M)

2 marks

State area rect − $\int$ — 1

Integrate $\int e^{2y}dx$ — 1

$\therefore$ Shaded area $= 4\times\ln 2 - \frac{3}{2}$ (M)

$= 4\ln 2 - 3/2$.

Qn	Solutions	Marks	Comments: Criteria
	Question 10		Knowledge of $\lim_{x \to 0} \frac{\sin x}{x} = 1$ is (1M)
a)	$\lim_{x \to 0} \frac{\sin 5x}{2x}$ $= \lim_{x \to 0} \frac{\sin 5x}{5x} \cdot \frac{5}{2}$ (1M) $= \frac{5}{2} \lim_{5x \to 0} \frac{\sin 5x}{5x}$ $= \frac{5}{2} \times 1$ (1M) $= \frac{5}{2}$		For 2 marks, $\frac{5}{2} \lim_{5x \to 0} \frac{\sin 5x}{5x}$ needs to be shown.
b)	$\triangle AOB$ is equilateral (1M) $\therefore \angle AOB = \frac{\pi}{3}$. Area of sector is $\frac{1}{2} \times 2^2 \times \frac{\pi}{3}$ $= \frac{2\pi}{3}$. (1M)		
(ii)	Area of the segment = Area of the sector − Area of the $\triangle AOB$ (1M) $= \frac{2\pi}{3} - \frac{1}{2} \times 2 \times 2 \times \sin \frac{\pi}{3}$ (1M) $= \frac{2\pi}{3} - 2 \times \frac{\sqrt{3}}{2} = \left(\frac{2\pi}{3} - \sqrt{3}\right)$ units^2.		

Qn	Solutions	Marks	Comments: Criteria

c) $y = 3\sin\left(x - \frac{\pi}{6}\right)$

Note $\quad y = 0$

$\sin\left(x - \frac{\pi}{6}\right) = 0$

$x - \frac{\pi}{6} = 0, \pi, 2\pi$

$x = \frac{\pi}{6}, \frac{7\pi}{6}.$ $\qquad$ (1M)

$\underline{x = 0}$ $\quad y = 3\sin(-\pi/6)$

$= -\frac{3}{2}.$

$x = 2\pi$
$y = 3\sin(2\pi - \pi/6)$
$= -\frac{3}{2}.$ $\qquad$ (1M)

endpts.

y is maximum, when

$\sin\left(x - \frac{\pi}{6}\right) = \cancel{\frac{\pi}{2}}\ 1,$

$x - \frac{\pi}{6} = \frac{\pi}{2}$

$x = \frac{2\pi}{3}.$ $\qquad$ ($\frac{1}{2}$M)

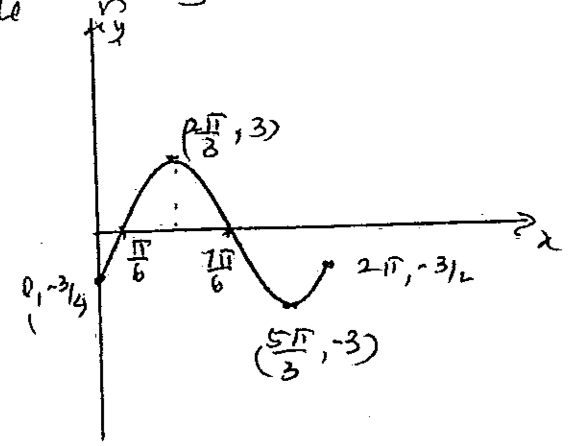
y is minimum when

$\sin\left(x - \frac{\pi}{6}\right) = -1$

$x - \frac{\pi}{6} = \frac{3\pi}{2}$

$x = \frac{5\pi}{3}$ $\qquad$ ($\frac{1}{2}$M)

amplitude is 3 $\quad$ Period is 2π



Qn	Solutions	Marks	Comments: Criteria
d)			

$y = x^2 e^x$

$y' = x^2 e^x + 2x e^x$

$\quad = x e^x (x+2)$ 1

At stationary point; $y' = 0$

$x e^x (x+2) = 0$

$x = 0$; $e^x \neq 0$; $x = -2$

$y = 0$ $\qquad y = (-2)^2 e^{-2}$

$\qquad\qquad = \frac{4}{e^2}$

$\therefore (0,0)$ and $\left(-2, \frac{4}{e^2}\right)$ are stationary points (1M)

Sign of y'

x	-0.1	0	0.1
y'	<0	0	>0

$\therefore (0,0)$ is a minimum turning point. (1M)

x	-2.1	-2	-1.9
y'	>0	0	<0

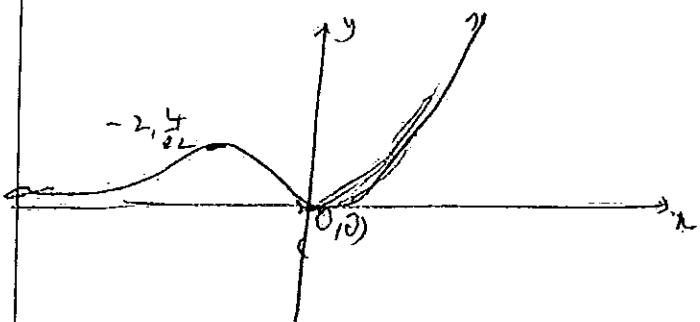
$\left(-2, \frac{4}{e^2}\right)$ is a maximum turning point. (1M)

$x \to \infty$; $y \to \infty$

$x \to -\infty$; $\quad y$: $x^2 \to \infty$

$\qquad e^x \to 0$

$\qquad \therefore y \to 0$ (1M)



(1M)

no c.o.e. marks for graph.

Qn	Solutions	Marks	Comments: Criteria

Question 11

a) $y = \ln \dfrac{(x-1)}{\sqrt{x}}$

if $\ln(x-1) - \ln(x^{\frac{1}{2}}) \to \dfrac{1}{x-1} - \dfrac{1}{\sqrt{x}} \times \dfrac{1}{2}x^{-\frac{1}{2}}$

$= \dfrac{1}{x-1} - \dfrac{1}{\sqrt{x}} \times \dfrac{1}{2\sqrt{x}}$

$= \ln(x-1) - \frac{1}{2}\ln x$

$y' = \dfrac{1}{x-1} - \dfrac{1}{2x} = \dfrac{2x-(x-1)}{2x(x-1)} = \dfrac{x+1}{2x(x-1)}$

b) $y = \dfrac{\sin x}{1+\cos x}$

$y' = \dfrac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$

$= \dfrac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

$= \dfrac{\cos x + 1}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$

$\therefore \displaystyle\int_0^{\pi/4} \frac{1}{1+\cos x}\,dx = \left(\frac{\sin x}{1+\cos x}\right)_0^{\pi/4}$ (1M)

$= \dfrac{\sin \pi/4}{1+\cos \pi/4} - 0$

$= \dfrac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} \xrightarrow{\text{clear steps needed}} = \dfrac{1}{\sqrt{2}+1} \cdot \dfrac{\sqrt{2}-1}{\sqrt{2}-1}$

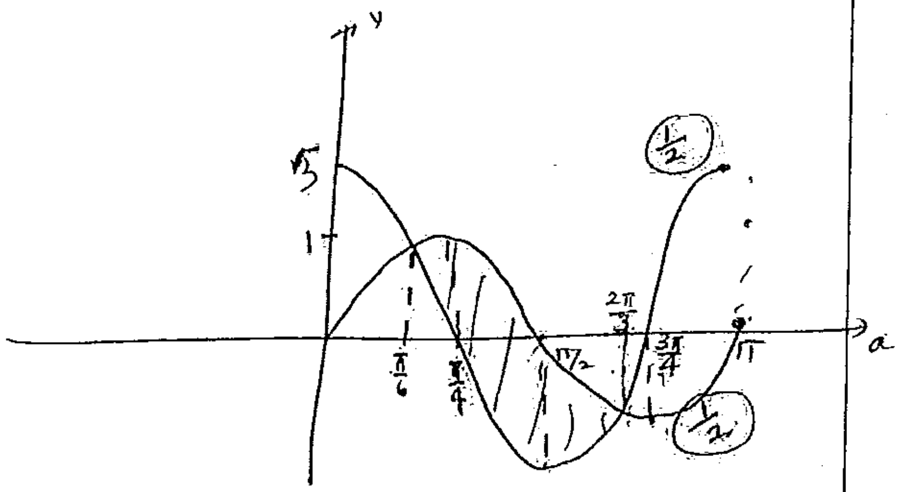
$= \sqrt{2} - 1$

rationalising needs to be shown clearly

1M

Qn	Solutions	Marks	Comments: Criteria

c) (i) $y = \sqrt{3}\cos 2x \;;\; y = \sin 2x$



(ii) $\sqrt{3}\cos 2x = \sin 2x$

$\tan 2x = \sqrt{3}$

$2x = \frac{\pi}{3}, \; \frac{4\pi}{3}$

$x = \frac{\pi}{6}, \; \frac{2\pi}{3}$.

correct limits (1)

(iii) Area $= \displaystyle\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \underbrace{\left(\sin 2x - \sqrt{3}\cos 2x\right)}_{\frac{1}{2}} dx$

$= \left(-\frac{\cos 2x}{2} - \sqrt{3}\,\frac{\sin 2x}{2} \right)_{\frac{\pi}{6}}^{\frac{2\pi}{3}}$ $\quad \frac{1}{2}$

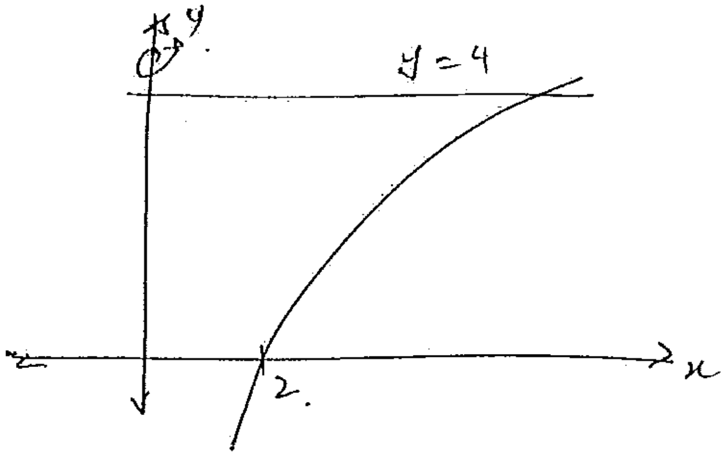
$= -\frac{1}{2}\left[\left(\cos \frac{4\pi}{3} - \cos \frac{\pi}{3}\right) + \sqrt{3}\left(\sin \frac{4\pi}{3} - \sin \frac{\pi}{3}\right) \right]$

$= -\frac{1}{2}\left[\left(-\frac{1}{2} - \frac{1}{2}\right) + \sqrt{3}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) \right]$

$= -\frac{1}{2}\left[(-1) + \sqrt{3}(-\sqrt{3}) \right]$

$= 2$. $\quad$ 1

Qn	Solutions	Marks	Comments: Criteria



$$V = \pi \int_0^4 x^2 \, dy$$

$$y = 2\ln(x-1)$$
$$\ln(x-1) = \frac{y}{2}$$
$$(x-1) = e^{y/2}$$
$$x = 1 + e^{y/2}.$$ (1M)

$$x^2 = 1 + e^y + 2e^{y/2}.$$

$$V = \pi \int_0^4 \left(1 + e^y + 2e^{y/2}\right) dy$$ (1M)

$$= \pi \left[y + e^y + \underbrace{2\,\frac{e^{y/2}}{1/2}}_{4e^{\frac{y}{2}}} \right]_0^4$$ (1M)

$$= \pi\left[(4 + e^4 + 4e^2) - (0 + 1 + 4e^0)\right]$$

$$= \pi\left[(e^4 + 4e^2 + 4) - 5\right]$$

$$= \pi\left(e^4 + 4e^2 - 1\right).$$ (1M)