SYDNEY GRAMMAR SCHOOL

2015 Half-Yearly Examination

FORM IV

MATHEMATICS

Tuesday 19th May 2015

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.

Total — 108 Marks

- All questions may be attempted.
- All necessary working should be shown.
- Start each question on a new page.

Collection

- Write your name, class and master on each page of your answers.
- Staple your answers in a single bundle.
- Write your name and master on this question paper and hand it in with your answers.

4A: KWM	4B: DNW	4C: NL	4D: SG
4E: MLS	4F: PKH	4G: FMW	4H: SJE
4I: SO	4J: DS		

Checklist

- Writing paper required.
- Candidature — 190 boys

Examiner
DNW

QUESTION ONE (12 marks) Start a new page.

(a) Write 345·6 in scientific notation.

(b) Write 4·893 correct to two significant figures.

(c) Write 7^{-2} as a fraction in simplest form.

(d) Simplify $2x - 4y + 6y - 5x$.

(e) Simplify $4\sqrt{3} + 6\sqrt{3}$.

(f) Expand $-3(2x - 4)$.

(g) Solve $2y + 3 = -7$.

(h) What is the y-intercept of the parabola with equation $y = x^2 - 3x + 4$?

(i) Solve $(x - 3)(x + 1) = 0$.

(j) What is the gradient of the line $y = 3x + 2$.

(k) Simplify $\sqrt{20}$.

(l) A parabola is concave up and its vertex is at $(2, 0)$.
How many x-intercepts does the parabola have?

Exam continues next page ...

QUESTION TWO (12 marks) Start a new page.

(a) Factorise $x^2 - 6x + 9$.

(b) Simplify $(4a^3)^2$.

(c) Write $\log_2 32 = 5$ in index form.

(d) Make x the subject of the formula $A = \frac{1}{2}xy$.

(e) Find the mid-point of the interval joining $P(-1, 2)$ and $Q(3, -1)$.

(f) Expand and simplify $(2x + 1)(x - 3)$.

(g) Simplify:

(i) $\sqrt{7} \times \sqrt{3}$

(ii) $\dfrac{\sqrt{30}}{\sqrt{5}}$

(h) Solve $x^2 - 7x + 12 = 0$.

QUESTION THREE (12 marks) Start a new page.

(a) Solve $\dfrac{x-2}{3} - \dfrac{x}{2} = \dfrac{1}{6}$.

(b) Simplify $\sqrt{50} - \sqrt{18}$.

(c) Express $3\sqrt{5}$ as the square root of a whole number.

(d) Express $\dfrac{1}{2\sqrt{3}}$ as a fraction with rational denominator.

(e) A parabola has intercepts at $x = -1$ and $x = 9$.
What is the equation of its axis of symmetry?

(f) Solve the quadratic equation $2x^2 - 3x - 5 = 0$.

(g) (i) Find the point where the line $2x - 3y = 1$ intersects the vertical line $x = 5$.

(ii) The horizontal line ℓ is concurrent with the two lines in part (i).
Find the equation of ℓ.

Exam continues overleaf ...

QUESTION FOUR (12 marks) Start a new page.

(a) (i) Find the gradient of the line through $P(1, 2)$ and $Q(-3, -1)$.

(ii) What is the gradient of a line perpendicular to PQ?

(iii) Find the equation of the line perpendicular to PQ which passes through $R(3, -2)$.
Give your answer in the form $y = mx + b$.

(b) Solve each quadratic equation using the method specified:

(i) $x^2 + 3x + 1 = 0$ by the quadratic formula,

(ii) $x^2 - 2x - 1 = 0$ by completing the square.

(c) A parabola has its vertex at $(1, 4)$ and y-intercept at $(0, 3)$. It has two x-intercepts, one of which is at $(3, 0)$.

(i) Is the parabola concave up or concave down? Give a reason for your answer.

(ii) The parabola has a second x-intercept. What is it?

(d) The value of x to two significant figures is 520.
Given that x is a whole number, what are the highest and lowest possible values of x?

QUESTION FIVE (12 marks) Start a new page.

(a) Expand and simplify $(2\sqrt{3} + 1)(\sqrt{3} - 2)$.

(b) Write the fraction $\dfrac{2\sqrt{5}}{\sqrt{5} - 1}$ with a rational denominator.

(c) Solve simultaneously:

$$x + 2y = 2$$
$$3x + 5y = 3$$

(d) Consider the parabola with equation $y = x^2 - 4x - 5$.

(i) State the concavity.

(ii) Write down the y-intercept.

(iii) Find the two x-intercepts.

(iv) Find the coordinates of the vertex.

(v) Sketch the parabola, showing all this information.

Exam continues next page ...

QUESTION SIX (12 marks) Start a new page.

(a) Solve:

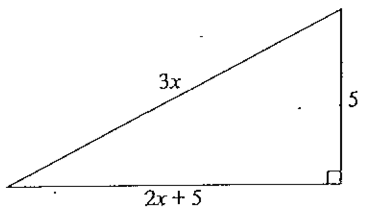
(i) $2^x = \dfrac{1}{16}$

(ii) $27^x = 81$

(b) Evaluate $\log_5 125$.

(c) Simplify $\dfrac{x^2 - x - 2}{2x^2 - x - 6}$.

(d)



The right-angled triangle in the diagram above has base $(2x + 5)$, altitude 5 and hypotenuse $3x$. Determine the exact value of x.

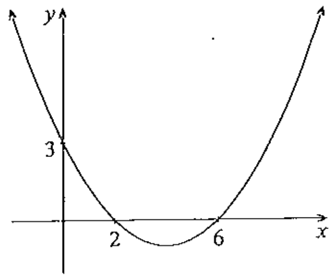
QUESTION SEVEN (12 marks) Start a new page.

(a) Solve $\log_x 36 = 2$.

(b) Express $\dfrac{1}{3x-2} - \dfrac{1}{3x+2}$ as a single fraction.

(c) Simplify $\dfrac{2x^2 - 3x - 2}{x^2 - 5x + 6} \times \dfrac{x}{2x+1}$.

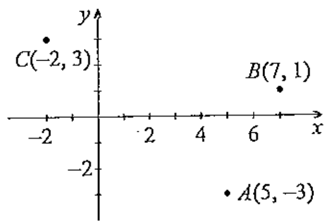
(d)



The graph above shows a parabola with x-intercepts at $(2, 0)$ and $(6, 0)$. Its y-intercept is at $(0, 3)$.

Determine the equation of the parabola. You may leave your answer in factored form.

(e)



The number plane above shows the points $A(5, -3)$, $B(7, 1)$ and $C(-2, 3)$.

(i) Find the distance CA.

(ii) A circle is drawn with centre C and radius CA. Show that this circle passes through B.

(iii) Using your answers to parts (i) and (ii), or otherwise, write 85 as the sum of two squares in two different ways.

QUESTION EIGHT (12 marks) Start a new page.

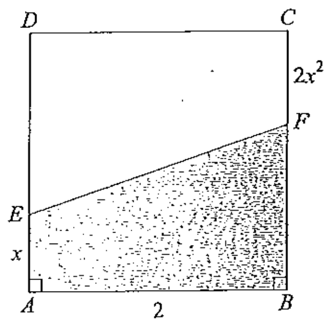
(a) Solve the equation $4^{2x+1} = 8^{x-1}$.

(b) Consider the equation $(x^2 - 2x)^2 - 7(x^2 - 2x) - 8 = 0$.

(i) Put $u = x^2 - 2x$ and solve the resulting quadratic equation for u.

(ii) Hence solve the original equation for x.

(c)



The diagram above shows a square $ABCD$ with side length 2 cm. The point E is on side AD with $AE = x$. The point F is on side BC with $CF = 2x^2$.

Let y be the area of the the shaded region $ABFE$.

(i) What range of values can x take?

(ii) Show that $y = 2 + x - 2x^2$.

(iii) Find the maximum area of $ABFE$.

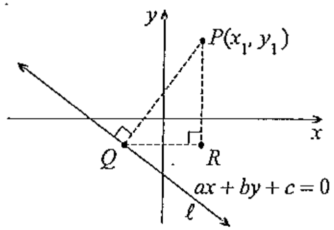
(iv) Find the minimum area of $ABFE$.

QUESTION NINE (12 marks) Start a new page.

(a) (i) Expand $(\sqrt{x} - \sqrt{y})^2$.

(ii) Using part (i), or otherwise, simplify $\sqrt{8 - 4\sqrt{3}}$.

(b)



The diagram above shows the line ℓ with equation $ax+by+c = 0$ and a point $P(x_1, y_1)$. The point Q is on ℓ and PQ is perpendicular to ℓ. The horizontal line through Q and the vertical line through P meet at R.

In the following, you may assume that the numbers a, b, c, x_1 and y_1 are all positive.

(i) (α) What is the gradient of ℓ?

(β) Show that $\dfrac{PR}{QR} = \dfrac{b}{a}$.

(ii) Since $\dfrac{PR}{QR} = \dfrac{b}{a}$, let $QR = \lambda a$ and $PR = \lambda b$ for some positive number λ.

(α) What are the coordinates of Q?

(β) Use the fact that Q lies on ℓ to help show that

$$PQ = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

(iii) (α) The derivation of the formula for PQ in parts (i) and (ii) is not valid when $a = 0$. Why not?

(β) Prove that the above formula for PQ is correct even when $a = 0$.

END OF EXAMINATION

QUESTION ONE (12 marks)

(a) $345{\cdot}6 = 3{\cdot}456 \times 10^2$ ☑

(b) $4{\cdot}893 \doteqdot 4{\cdot}9$ (to two significant figures.) ☑

(c) $7^{-2} = \frac{1}{49}$ ☑

(d) $2x - 4y + 6y - 5x = 2y - 3x$ ☑

(e) $4\sqrt{3} + 6\sqrt{3} = 10\sqrt{3}$ ☑

(f) $-3(2x - 4) = 12 - 6x$ ☑

(g) $2y + 3 = -7$
$2y = -10$
$y = -5$ ☑

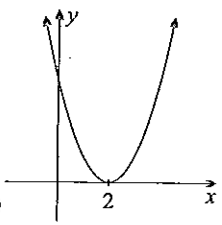
(h) The y-intercept of $y = x^2 - 3x + 4$ is $(0, 4)$. ☑
[Accept $y = 4$ or just 4]

(i) $(x - 3)(x + 1) = 0$
so $x = -1$ or 3 ☑

(j) The gradient of $y = 3x + 2$ is 3. ☑

(k) $\sqrt{20} = 2\sqrt{5}$ ☑

(l) The parabola has one x-intercept (which is the vertex.) ☑



Total for Question 1: 12 Marks

QUESTION TWO (12 marks)

(a) $x^2 - 6x + 9 = (x - 3)^2$ ☑

(b) $(4a^3)^2 = 16a^6$ ☑

(c) $\log_2 32 = 5$ is equivalent to $2^5 = 32$. ☑

(d) $A = \frac{1}{2}xy$ so $x = \dfrac{2A}{y}$ ☑

(e) $$M = \left(\frac{-1+3}{2}, \frac{2+-1}{2}\right)$$ ☑
$$= (1, \tfrac{1}{2})$$ ☑

[An answer with no working gets all or nothing.]

(f) $$(2x+1)(x-3) = 2x^2 - 6x + x - 3$$ ☑
$$= 2x^2 - 5x - 3$$ ☑

[An answer with no working gets all or nothing.]

(g) (i) $\sqrt{7} \times \sqrt{3} = \sqrt{21}$ ☑

(ii) $\dfrac{\sqrt{30}}{\sqrt{5}} = \sqrt{6}$ ☑

(h) $$x^2 - 7x + 12 = 0$$
$$(x-4)(x-3) = 0$$ ☑
so $x = 3$ or 4 ☑

Total for Question 2: 12 Marks

QUESTION THREE (12 marks)

(a) $$\frac{x-2}{3} - \frac{x}{2} = \frac{1}{6}$$
thus $2(x-2) - 3x = 1$ ☑
so $-x - 4 = 1$
hence $x = -5$ ☑

(b) $\sqrt{50} - \sqrt{18} = 5\sqrt{2} - 3\sqrt{2}$ ☑
$= 2\sqrt{2}$ ☑

(c) $3\sqrt{5} = \sqrt{45}$ ☑

(d) $$\frac{1}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$ ☑
$$= \frac{\sqrt{3}}{6}$$ ☑

(e) The axis is $x = \dfrac{-1+9}{2}$ (the average of the x-intercepts)
so, the line $x = 4$ ☑

(f) $$2x^2 - 3x - 5 = 0$$
$$(2x-5)(x+1) = 0$$ ☑
thus $x = -1$ or $\frac{5}{2}$ ☑

(g) (i) At $x = 5$, $10 - 3y = 1$
so $y = 3$ ☑
That is, the point of intersection is $(5, 3)$.

(ii) The horizontal line through $(5, 3)$ is $y = 3$ ☑

Total for Question 3: 12 Marks

QUESTION FOUR (12 marks)

(a) (i) Gradient of $PQ = \dfrac{2+1}{1+3}$
$= \dfrac{3}{4}$ ☑

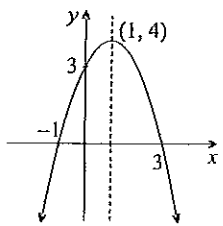
(ii) Perpendicular gradient $= -\frac{4}{3}$ ☑

(iii) Hence the equation of the line is
$y + 2 = -\frac{4}{3}(x-3)$ ☑
thus $y = -\frac{4}{3}x + 2$ ☑

(b) (i) $x^2 + 3x + 1 = 0$
so $b^2 - 4ac = 5$ ☑
thus $x = \dfrac{-3-\sqrt{5}}{2}$ or $\dfrac{-3+\sqrt{5}}{2}$ ☑

(ii) $x^2 - 2x - 1 = 0$
so $x^2 - 2x + 1 = 2$ ☑
or $(x-1)^2 = 2$
thus $x = 1 - \sqrt{2}$ or $1 + \sqrt{2}$ ☑

(c) (i) The parabola is concave down ☑
since the vertex is above the x-intercept ☑
[or any other valid argument.]



(ii) The other x-intercept is $(-1, 0)$ by symmetry. ☑
[accept $x = -1$ or just -1.]

(d) Lowest $x = 515$ and highest $x = 524$. ☑

Total for Question 4: 12 Marks

QUESTION FIVE (12 marks)

(a) $(2\sqrt{3}+1)(\sqrt{3}-2) = 2\times 3 - 4\sqrt{3} + \sqrt{3} - 2$ ☑

$= 4 - 3\sqrt{3}$ ☑

[An answer with no working gets all or nothing.]

(b) (i) $\dfrac{2\sqrt{5}}{\sqrt{5}-1} = \dfrac{2\sqrt{5}}{\sqrt{5}-1} \times \dfrac{\sqrt{5}+1}{\sqrt{5}+1}$ ☑

$= \dfrac{10+2\sqrt{5}}{5-1}$

$= \dfrac{5+\sqrt{5}}{2}$ ☑

(c) $x + 2y = 2$ (1)

$3x + 5y = 3$ (2)

Multiply (1) by three to get

$3x + 6y = 6$ (3) ☑

Now subtract (2) from (3)

$y = 3$ ☑

and from (1) the value of x is

$x = -4$ ☑

(d) $y = x^2 - 4x - 5$

$= (x-5)(x+1)$

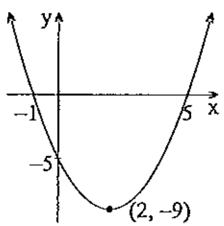
(i) The parabola is concave up ($a = 1$) ☑

(ii) $(0, -5)$ or $y = -5$ or just -5. ☑

(iii) $x = -1$ or 5 ☑

(iv) $(2, -9)$ ☑

(v) All or nothing. ☑



Total for Question 5: 12 Marks

QUESTION SIX (12 marks)

(a) (i) $2^x = \dfrac{1}{16}$

$= 2^{-4}$

so $x = -4$ ☑

(ii) $27^x = 81$

thus $3^{3x} = 3^4$ ☑

so $3x = 4$

hence $x = \frac{4}{3}$ ☑

(b) $\log_5 125 = \log_5 (5^3)$ ☑

$= 3$ ☑

(c) $\dfrac{x^2 - x - 2}{2x^2 - x - 6} = \dfrac{(x-2)(x+1)}{(2x+3)(x-2)}$ ☑☑

$= \dfrac{x+1}{2x+3}$ ☑

(d) $(3x)^2 = (2x+5)^2 + 5^2$ (by Pythagoras) ☑

Re-arrange this to get

$5x^2 - 20x - 50 = 0$

so $x^2 - 4x - 10 = 0$ ☑

for which $b^2 - 4ac = 56$

so $x = \dfrac{4 \pm \sqrt{56}}{2}$. ☑

Now simplify and use the fact that x must be positive to get

$x = 2 + \sqrt{14}$ ☑

Total for Question 6: 12 Marks

QUESTION SEVEN (12 marks)

(a) $\log_x 36 = 2$

so $x^2 = 36$ ☑

thus $x = 6$ $(x > 0)$ ☑

[Penalise the answer $x = -6$.]

(b) $\dfrac{1}{3x-2}-\dfrac{1}{3x+2}=\dfrac{(3x+2)-(3x-2)}{(3x-2)(3x+2)}$ √

$=\dfrac{4}{(3x-2)(3x+2)}$ √

(c) $\dfrac{2x^2-3x-2}{x^2-5x+6}\times\dfrac{x}{2x+1}=\dfrac{(2x+1)(x-2)}{(x-3)(x-2)}\times\dfrac{x}{2x+1}$ √√

$=\dfrac{x}{x-3}$ √

(d) Using the x-intercepts, for some number a,

$y=a(x-2)(x-6)$ √

At $x=0$, $3=12a$

so $a=\frac{1}{4}$ √

thus $y=\frac{1}{4}(x-2)(x-6)$

(e) (i) $CA^2=(5+2)^2+(-3-3)^2$

so $CA=\sqrt{85}$ √

(ii) $CB^2=(7+2)^2+(1-3)^2$

$=85$

$=CA^2$ √

Hence both A and B are on the circle with centre C and radius $\sqrt{85}$.

(iii) $85=7^2+6^2=9^2+2^2$ √

Total for Question 7: 12 Marks

QUESTION EIGHT (12 marks)

(a) $4^{2x+1}=8^{x-1}$

or $2^{4x+2}=2^{3x-3}$ √

so $4x+2=3x-3$ √

thus $x=-5$ √

(b) $(x^2-2x)^2-7(x^2-2x)-8=0$

(i) Put $u=x^2-2x$ to get

$u^2-7u-8=0$ √

$(u-8)(u+1)=0$

thus $u=-1$ or 8 √

(ii) When $u=-1$: $x^2-2x+1=0$

or $(x-1)^2=0$

so $x=1$ √

When $u=8$: $x^2-2x-8=0$

or $(x-4)(x+2)=0$

so $x=-2$ or 4 √

(c) (i) From side BC and length FC,

$0\le 2x^2\le 2$

so $0\le x\le 1$ $(x\ge 0)$ √

(ii) The formula for the area of a trapezium is $A=\frac{1}{2}(a+b)h$. Thus:

$y=\frac{1}{2}\Big(x+(2-2x^2)\Big)\times 2$ √

$=2+x-2x^2$

(iii) This is a concave down parabola, hence the maximum will occur at the vertex,

where $x=\frac{1}{4}$ (by the formula $x=-\frac{b}{2a}$) √

and $y_{\max}=2+\frac{1}{4}-2\times\frac{1}{16}$

$=2\frac{1}{8}$ or $\frac{17}{8}$. √

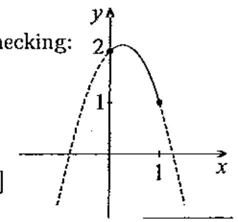
(iv) The minimum may occur at the extreme values of x. Checking:

at $x=0$, $y=2$

at $x=1$, $y=2+1-2=1$

hence $y_{\min}=1$ √

[Note that $ABFE$ is in fact a triangle when $x=0$ or 1.]



Total for Question 8: 12 Marks

QUESTION NINE (12 marks)

(a) (i) $(\sqrt{x}-\sqrt{y})^2=x+y-2\sqrt{xy}$ √

(ii) Let $x+y-2\sqrt{xy}=8-2\sqrt{12}$

then $x+y=8$ (1)

$xy=12$ (2) √

From (1), $y=8-x$ so in (2)

$x(8-x)=12$

or $x^2-8x+12=0$.

Now $(x-6)(x-2)=0$

thus $x=6$ or 2 √

for which $y=2$ or 6.

[Award the mark for a guess.]

So $\quad 8 - 4\sqrt{3} = (\sqrt{6} - \sqrt{2})^2$

hence $\sqrt{8 - 4\sqrt{3}} = \sqrt{6} - \sqrt{2}$ ☑

(b) (i) (α) The line $ax + by + c = 0$ has gradient $-\frac{a}{b}$. ☑

(β)
$$\begin{aligned}\frac{PR}{QR} &= \text{gradient of } PQ\\ &= -1 \div \text{gradient of } \ell\\ &= \frac{b}{a}\end{aligned}$$ ☑

(ii) (α) $\quad Q = (x_1 - \lambda a, y_2 - \lambda b)$ ☑

(β) Since Q is on ℓ, substitute the coordinates of Q to get:
$a(x_1 - \lambda a) + b(y_1 - \lambda b) + c = 0$. ☑

Rearranging this:
$$ax_1 + by_1 + c = \lambda(a^2 + b^2)$$

thus $\quad \lambda = \dfrac{ax_1 + by_1 + c}{a^2 + b^2}$. ☑

Next
$$\begin{aligned}PQ^2 &= QR^2 + RP^2\\ &= \lambda^2(a^2 + b^2)\end{aligned}$$

so
$$\begin{aligned}PQ &= \lambda\sqrt{a^2 + b^2}\\ &= \frac{ax_1 + by_1 + c}{a^2 + b^2} \times \sqrt{a^2 + b^2}\\ &= \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.\end{aligned}$$ ☑

(iii) (α) When $a = 0$, the value of $\dfrac{PR}{QR} = \dfrac{b}{a}$ is undefined. ☑

[There are other possible explanations.]

(β) When $a = 0$ the line ℓ is horizontal with equation $y = -\frac{c}{b}$.
Hence PQ is just the difference in the y-coordinates.
That is $PQ = y_1 + \frac{c}{b}$.
The formula for PQ gives:
$$\begin{aligned}PQ &= \frac{0 + by_1 + c}{\sqrt{0 + b^2}}\\ &= \frac{by_1 + c}{b}\\ &= y_1 + \tfrac{c}{b} \qquad \text{(exactly as it should!)}\end{aligned}$$ ☑

Total for Question 9: 12 Marks

DNW