Alexandria Park Community School

2016 HIGHER SCHOOL CERTIFICATE
HALF YEARLY EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 8-11, show relevant mathematical reasoning and/or calculations

Total Marks – 67

Section I

7 marks

- Attempt questions 1-7
- Allow about 12 minutes for this section

Section II

60 marks

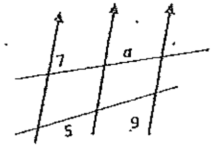
- Attempt questions 8-11
- Allow about 1 hour and 48 minutes for this section

Section I

7 marks
Attempt Questions 1-7
Allow about 12 minutes for this section

Use the multiple-choice answer sheet for Questions 1-7.

1

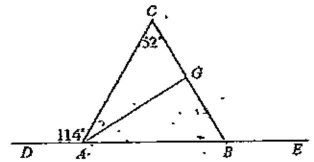


Not to scale

The value of a in the above diagram is:

(A) 9 (B) 11 (C) 12 (D) 12.6

2 In the diagram, $\angle CAD = 114°$ and $\angle ACB = 52°$. DE is a straight line. AG bisects $\angle CAB$.

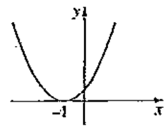


What is the value of $\angle AGB$?

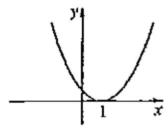
(A) 33° (B) 52° (C) 62° (D) 85°

3 Which graph best represents $y = x^2 + 2x + 1$?

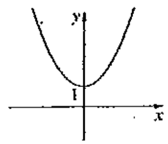
(A)



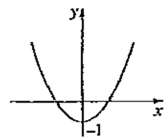
(B)



(C)



(D)



4 What is the solution to the equation $\log_3(x+1) = 4$?

(A) 11 (B) 81 (C) 80 (D) 12

5 Which equation represents the line parallel to $2x - 3y = 8$, passing through the point $(1, 2)$?

(A) $3x + 2y - 1 = 0$ (B) $3x + 2y - 8 = 0$

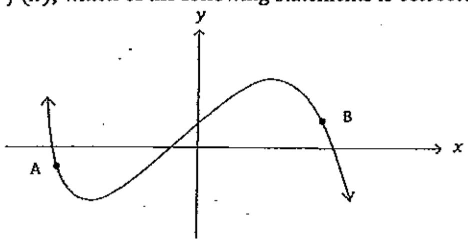
(C) $2x - 3y - 8 = 0$ (D) $2x - 3y + 4 = 0$

6 The correct solutions to the equation $2\sin^2 x - 1 = 0$ for $-\pi \le x \le \pi$ are

(A) $\frac{\pi}{4}, \frac{3\pi}{4}$ (B) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(C) $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ (D) $\pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

7 For the curve $y = f(x)$, which of the following statements is correct?



(A) $f'(x) > 0$ at A and $f''(x) < 0$ at B

(B) $f'(x) < 0$ at A and $f''(x) < 0$ at B

(C) $f'(x) > 0$ at A and $f''(x) > 0$ at B

(D) $f'(x) < 0$ at A and $f''(x) > 0$ at B

End of Section I

Section II

60 marks
Attempt Questions 8-11
Allow about 1 hour and 48 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 8-11, your responses should include relevant mathematical reasoning and/or calculations.

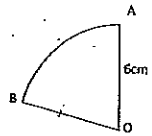
Question 8 (15 marks) Use the Writing Booklet.

(a) Differentiate $\frac{x+1}{x^2}$ **2**

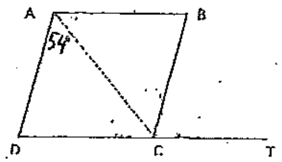
(b) Find $\int \frac{dx}{(2x+1)^3}$ **2**

(c) Find $\int \cos\frac{x}{2}\, dx$ **2**

(d) The perimeter of the sector AOB is 15cm. Calculate the size of angle AOB, correct to the nearest degree. **2**



(e) In the diagram, $ABCD$ is a rhombus where $\angle DAC = 54°$ and DC is produced to T.

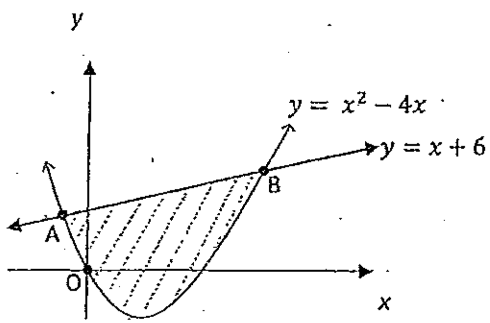


Copy the diagram into your booklet.

(i) What is the value of $\angle DAB$? **1**

(ii) What is the value of $\angle BCT$? **1**

(f) The parabola $y = x^2 - 4x$ and the line $y = x + 6$ intersect at the points A and B.



(i) Find the x-coordinate of the points A and B. **2**

(ii) Calculate the area enclosed by the parabola $y = x^2 - 4x$ and the line $y = x + 6$ **3**

End of Question 8

Question 9 (15 marks) Use the Writing Booklet.

(a) (i) Write down the exact value of $\tan 2x$ when $x = \frac{\pi}{6}$ **1**

(ii) Give the exact value of $\int_0^{\frac{\pi}{6}} \sec^2 2x \; dx$ **2**

(b) Simplify the expression $\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\sin(\pi+\theta)}$ **2**

(c) Consider the curve $y = 2x^3 + 3x^2 - 12x - 9$

(i) Find the coordinates of any stationary points and determine their nature. **3**

(ii) Show that a point of inflexion exists and state its coordinates. **2**

(iii) Sketch the curve $y = f(x)$ in the domain $-3 \leq x \leq 3$, showing the y-intercept. **2**

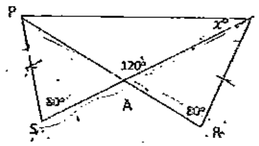
(iv) For what values of x, in the domain given in part (iii), is the curve both increasing and concave down? **2**

(v) Write down the minimum value for $y = f(x)$ in the interval $-3 \leq x \leq 3$. **1**

End of Question 9

Question 10 (15 marks) Use the Writing Booklet.

(a)



NOT TO SCALE

PR and QS are straight lines intersecting at a point A. Also $PS = QR$, $\angle PSA = \angle QRA = 80°$, $\angle PAQ = 120°$ and $\angle PQA = x$.

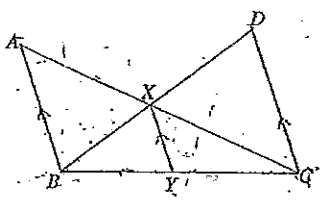
(i) Copy the diagram into your booklet.

(ii) Prove that ΔPSA is congruent to ΔQRA **3**

(iii) Hence, prove that ΔPAQ is isosceles and find the value of x. **3**

(b) Find the value/s of m required for the line $y = mx - 12$ to be a tangent to the parabola $y = 2x^2 - x - 10$ **3**

(c) In the diagram below $AB||XY||DC$.



NOT TO SCALE

(i) Copy the diagram into your booklet

(ii) Prove that ΔAXB is similar to ΔCXD **2**

(iii) If $XB = 12cm, XC = 30cm, BY = 8cm$ and $YC = 24cm$, find the length of AX and DX, giving reasons. **3**

(iv) Hence, find $AB : DC$ **1**

End of Question 10

Question 11 (15 marks) Use the Writing Booklet.

(a) (i) Differentiate $2xe^{-x}$ **2**

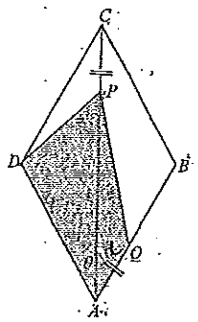
(ii) Hence find $\int_0^1 xe^{-x}dx$ **2**

(b) The region bounded by the curve $y = 1 + x^2$ and the x-axis between $x = 0$ and $x = 3$ is rotate about the x-axis to form a solid. Find the volume of the solid. **3**

(c) $ABCD$ is a rhombus with $2cm$ sides.

P and Q are points on AC and AB respectively such that $CP = AQ = x\ cm$. $\angle DAP = \theta$ (where $0 < \theta < \frac{\pi}{2}$) and θ is constant.

Let the shaded area $PDAQ$ be equal to S cm^2.



(i) Show that $S = \frac{\sin\theta}{2}(4\cos\theta - x)(2 + x)$ **3**

(ii) If $\frac{dS}{dx} = 0$, find x in terms of θ. **2**

(iii) Find $\frac{d^2S}{dx^2}$ in terms of θ **1**

(iv) Suppose that $\theta = \frac{\pi}{6}$, show that S attains its maximum value when $\frac{PC}{AC} = \frac{\sqrt{3}-1}{2\sqrt{3}}$ **2**

End of paper

2016 Alexandria park

HSC Half yearly examination. Pg. 1

1. $\frac{9}{5} \times 7 = 12.6$ (D)

2. $\angle CAB = 180° - 114° = 66°$ ($\angle$ sum of straight line)

$\angle CAG = \angle GAB = 33°$ (since AG Bisects $\angle CAB$)

$\therefore \angle CGA = 180° - 52° - 33° = 95°$ ($\angle$ sum of triangle)

$\therefore \angle AGB = 180° - 95° = 85°$ ($\angle$ sum of straight line)

(D)

3. $y = x^2 + 2x + 1$

what happens at $y=0$?

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$ (A)

i.e a double root at $x = -1$

4. $\log_3(x+1) = 4$

$(x+1) = 3^4$

$(x+1) = 81$

$x = 80$ (C)

5. parallel to $2x - 3y = 8$

that means the new line must have the same gradient.

(1.) make into general form $y = mx + b$

$3y = 2x - 8$

$y = \frac{2}{3}x - \frac{8}{3}$ $m = \frac{2}{3}$.

(2) point gradient formula

$(y - y_1) = m(x - x_1)$

at $(1, 2)$

$y - 2 = \frac{2}{3}(x - 1)$

$3y - 6 = 2x - 2.$ (D)

$\therefore 2x - 3y + 4 = 0$

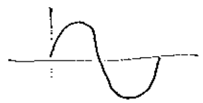
Pg 2

6. $2\sin^2 x - 1 = 0.$

i.e. $\sin^2 x = \frac{1}{2}.$

$\sin x = \pm\frac{1}{\sqrt{2}}.$

So it is in all 4 quadrants.



S	A
T	C.

$= \frac{\pi}{4} + \frac{2k\pi}{4}$ for $0 \leq k \leq 3$ $k \in \mathbb{Z}$

for $0 \leq x \leq 2\pi$

$= \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ for $-\pi \leq x \leq \pi$

$=$ (C)

7. (B.)

8. Differentiate $\frac{x+1}{x^2}$

$= \frac{1}{x} + \frac{1}{x^2} = x^{-1} + x^{-2}$

$= -x^{-2} - 2x^{-3}.$

$= \frac{-1}{x^2} - \frac{2}{x^3} = \frac{-x-2}{x^3}$

$= \frac{-(x+2)}{x^3}.$

Pg (3)

b) $\int \frac{dx}{(2x+1)^3}$

$= \int (2x+1)^{-3}\,dx$

$= \left[\frac{(2x+1)^{-2}}{-4}\right] + C$

c) $\int \cos\frac{x}{2}\,dx$

$= \int \cos(\tfrac{1}{2}x)\,dx$

$= 2\sin\frac{x}{2} + C$

d) AOB = 15 cm.

AO = OB = 6cm (Radius of circle)

∴ AB = 15 - 6 - 6 = 3cm.

$L = \theta \times r$ (θ in radians)

$\theta = \frac{L}{r} =$

e) AB = BC = CD = DA.

All rhombuses are parallelograms

∴ ∠ACB = 54°

∠CAB = 54° (Diagonals Bisect vertices in a Rhombus)

∴ ∠DAB = ∠CAB + ∠DAC

= 54° + 54° = 108°

ii) ∠ACB = 54° (Alt. ∠)

∠ACD = 54° (Diagonals Bisect)

∴ ∠BCD = 108° = ∠DAB.

∴ ∠BCT = 180° - 108°

= 72° (∠ sum of straight line)

f) i) Simultaneous equations:

$x^2 - 4x = x + 6$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

Points are $x = 6, x = -1$

A: -1, B: 6 for x co-ord

we now have our integration limits

A: $\int_{-1}^{6} x+6\,dx - \int_{-1}^{6} (x^2-4x)\,dx$

$= \int_{-1}^{6} x+6-(x^2-4x)\,dx$

$= \int_{-1}^{6} x+6-x^2+4x\,dx$

$= \int_{-1}^{6} -x^2+5x+6\,dx$

$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} + 6x\right]_{-1}^{6}$

$= \left[-\frac{216}{3} + \frac{180}{2} + 36\right] - \left[\frac{1}{3} + \frac{5}{2} - 6\right]$

$= -72 + 90 + 36 - \left[\frac{2}{6} + \frac{15}{6} - \frac{36}{6}\right]$

$= 54 - \left[\frac{-19}{6}\right]$

$= \frac{324}{6} + \frac{19}{6} = \frac{343}{6}$ units^2

9. i) $\tan 2x$ when $x = \frac{\pi}{6}$

$= \tan\frac{\pi}{3} = \sqrt{3}$.

ii) $\int_0^{\pi/6} \sec^2 2x\,dx$

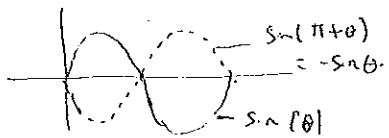
$= \left[\frac{1}{2}\tan(2x)\right]_0^{\pi/6}$

$= \frac{1}{2} \times \sqrt{3} = \frac{\sqrt{3}}{\ }$

Pg (4)

b) $\frac{\cos(\frac{\pi}{2}-\theta)}{\sin(\theta+\pi)}$

$= \frac{\sin\theta}{-\sin\theta} = -1$



c). $y = 2x^3 + 3x^2 - 12x - 9$

i) $\frac{dy}{dx} = 6x^2 + 6x - 12$.

$6(x^2+x-2) = 0$ to find stat. points

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

1, -2 are stat. points

$\frac{d^2y}{dx^2} = 12x + 6$.

when $x = -2$, $y'' = -18$

$-18 < 0$ ∴ max

when $x = 1$, $y'' = +ve = 18$

$18 > 0$ ∴ min.

stationary pts

(-2, 11) max

(1, -16) min

ii) Point of inflexion.

$$\frac{d^2y}{dx^2} = 12x - 6$$

let $y'' = 0$ to find point of inflexion

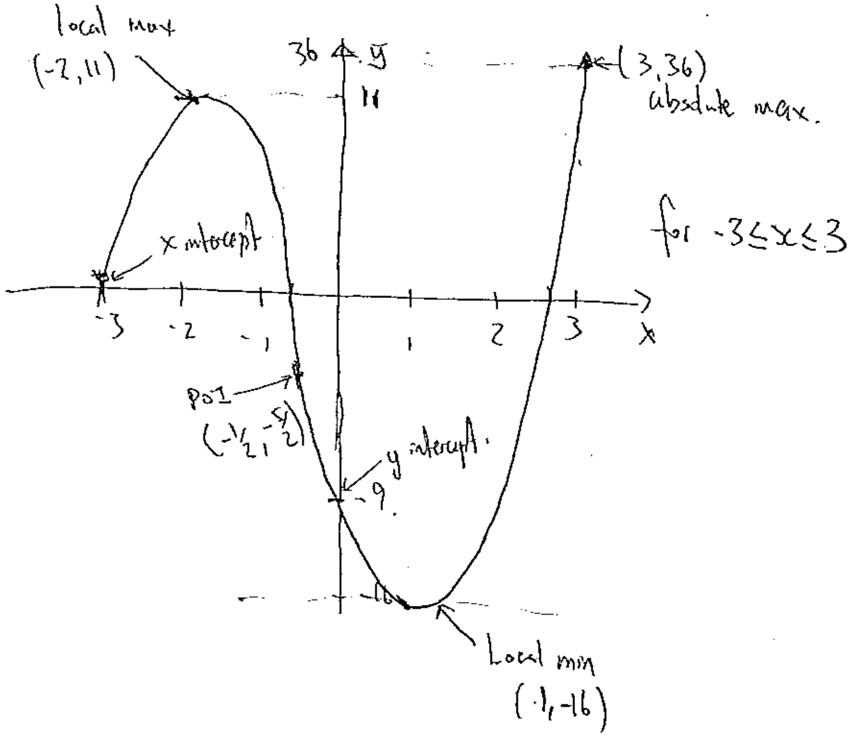
i.e. $12x - 6 = 0$

$12x = 6$

$x = \frac{6}{12} = \frac{1}{2}$

at $x = \frac{1}{2}$, $y = -\frac{5}{2}$

iii)



for $-3 \leq x \leq 3$.

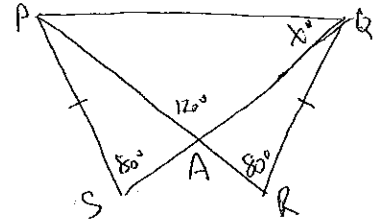
iv) concave down and increasing between $-3 \leq x \leq -2$.

v) min value for $y = f(x)$ in $-3 \leq x \leq 3 = -16$.

10.

a)

i)



ii) $\angle PSA = \angle QRA$ (given)

$PS = QR$ (given)

$\angle PAS = \angle QAR$ (vertically opp.)

$\therefore \triangle PSA \equiv \triangle QRA$ (AAS)

iii) $PA = AQ$ (corresponding sides of a congruent triangle are equal)

ie $120° + 2x = 180°$

$2x = 60°$

$x = 30°$

b) $y_1 = 2x^2 - x - 10$

$y_2 = mx - 12$.

So the line will only hit the parabola at one point. That is to say $y_1 = y_2$ has only one soln.

$2x^2 - x - 10 = mx - 12$ has ONE solution.

HOW DO WE FIND OUT?

USE DISCRIMINANT.

$2x^2 - x - mx + 2 = 0$

$2x^2 - x(1+m) + 2 = 0$

$a = 2$, $b = -(1+m)$ $c = 2$

$\Delta = b^2 - 4ac = 0$

$(m^2 + 1 + 2m) - 4(4) = 0$ This satisfies tangent

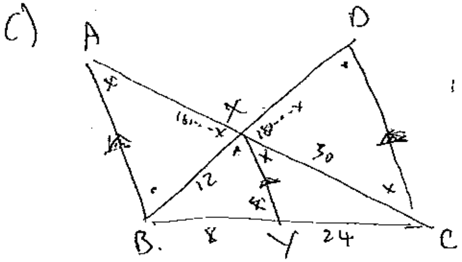
$m^2 + 1 + 2m - 16 = 0$

$m^2 + 2m - 15 = 0$

$(m+5)(m-3) = 0$

ie $m = -5$ OR $m = 3$.

c)



i) $\angle BAC = \angle ACD$ (alternate $\angle$'s)

$\angle CDB = \angle DBA$ (")

$\therefore \Delta AXB \,|||\, \Delta CXD$ (equi-Angular)

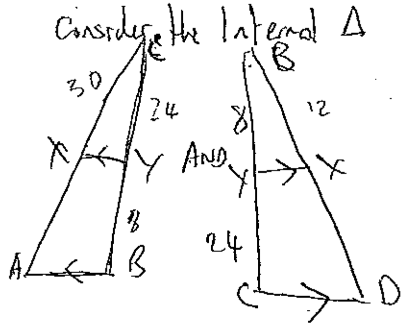
ii) $XB = 12\text{cm}$

$XC = 30\text{cm}$

$BY = 8\text{cm}$

$\frac{AX}{BX} = \frac{CX}{DX}$ (Since $\Delta AXB \,|||\, \Delta CXD$)

Consider the Internal Δ



iii) using ratios

$\frac{30}{30+AX} = \frac{24}{24+8}$, $\frac{8}{8+24} = \frac{12}{12+DX}$

$AX = 10$ $DX = 36$

iv) $AB : DC = 1:3$

Pg 7

11 a)

i) $2xe^{-x}$

let $u = 2x$, $u' = 2$

$v = e^{-x}$ $v' = -e^{-x}$

$\frac{d}{dx} = u'v + v'u$

$= 2e^{-x} + (-2xe^{-x})$

$= 2e^{-x} - 2xe^{-x}$

$= 2e^{-x}(1-x)$

ii) from part i)

$\int 2e^{-x} - 2xe^{-x} = 2xe^{-x}$

$\int 2xe^{-x} = \left[\int 2e^{-x}\right] - 2xe^{-x}$

$2\int_0^1 xe^{-x} = 2\left[-e^{-x}\right] - 2xe^{-x}$

$\int_0^1 xe^{-x} = \left[-e^{-x} - xe^{-x}\right]_0^1$

$= \left[-e^{-x}(1+x)\right]_0^1$

$= -e^{-1}(2) + e^{0}(1)$

$= -2e^{-1} + 1$

Pg 8

b) $y = 1 + x^2$

$x = 0$ to $x = 3$.

$V = \pi \int_0^3 y^2 \, dx$

$= \pi \int_0^3 1 + x^4 + 2x^2 \, dx$

$= \pi \left[x + \frac{x^5}{5} + \frac{2x^3}{3} \right]_0^3$

$= \pi \left[3 + \frac{243}{5} + \frac{2(27)}{3} \right]$

$= \pi \left[\frac{348}{5} \right]$

$= \frac{348\pi}{5} u^3.$

c) $CP = AQ = x$ cm.

$\angle DAP = \theta$ $[0 < \theta < \frac{\pi}{2}]$

$AD = DC = CB = AB = 2\text{cm}$

using area formula for triangles.

$[\frac{1}{2} ab \sin C]$

$=$ Area ΔDPA + Area ΔPAQ

$=$ Area $PDAQ$ $(PA = CA - x)$

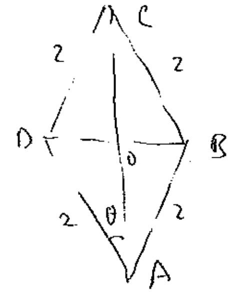
in ΔDPA

Area $= \frac{1}{2}(2)(PA) \sin\theta$

How do we solve for PA?

CONSTRUCTION!

1. CONSTRUCT DB and AC. Intersection is the mid points of both lines. Lets call this 'O'



$\frac{AO}{DA} = \cos\theta \implies AO = 2\cos\theta$

So $CA = 2AO = 2[2\cos\theta] = 4\cos\theta$

Therefore $PA = CA - x$

$= 4\cos\theta - x.$

in ΔPAQ

Area $= \frac{1}{2}(x)(4\cos\theta - x)\sin\theta$

Since $\angle DAC = \angle CAB$ (property of rhombus)

$\therefore S = \frac{\sin\theta}{2}(4\cos\theta - x)(2+x)$

ii) $\frac{dS}{dx} = 0.$

$\frac{d}{dx}\left[2\sin\theta\cos\theta - \frac{x\sin\theta}{2}\right]\left[2+x\right] = 0$

$4\sin\theta\cos\theta + 2x\sin\theta\cos\theta - x\sin\theta - \frac{x^2\sin\theta}{2} = 0$

$\frac{1}{x}$ $4\sin\theta\cos\theta + 2x\sin\theta\cos\theta - x\sin\theta - \frac{x^2\sin\theta}{2} = 0.$

$0 + 2\sin\theta\cos\theta - \sin\theta - x\sin\theta = 0$

$2\sin\theta\cos\theta - \sin\theta = x\sin\theta$

$x = 2\cos\theta - 1$

iii) $\frac{ds}{dx} = 2\sin\theta\cos\theta - \sin\theta - x\sin\theta$

$\frac{d^2s}{dx^2} = -\sin\theta$

iv) at $\theta = \frac{\pi}{6}$

$\frac{PC}{AC} = \frac{x}{4\cos\theta}$

max at $x = 2\cos\theta - 1.$

$= 2\cos(\frac{\pi}{6}) - 1$

$= 2(\frac{\sqrt{3}}{2}) - 1$

$= \sqrt{3} - 1$

$4\cos\theta = 4(\frac{\sqrt{3}}{2})$

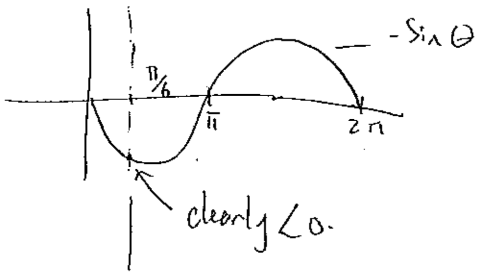
$= 2\sqrt{3}$

ie. $\frac{PC}{AC} = \frac{x}{4\cos\theta} = \frac{\sqrt{3}-1}{2\sqrt{3}}.$

pg 9.

Proof of maximum.

$\frac{d^2s}{dx^2} < 0$



$\therefore$ maximum.