Centre Number

CRANBROOK SCHOOL

Student Number

2016

HSC Trial Examination

Assessment Task 3

Mathematics Extension 2

Reading time	5 minutes
Writing time	3 Hours
Total Marks	~~100~~ 98
Task weighting	40%

General Instructions

- Write using a black pen
- Diagrams drawn using pencil
- A Board-approved calculator may be used
- A Board of Studies Reference Sheet can be found at the end of this paper
- Use the Multiple-Choice Answer Sheet provided
- All relevant working should be shown for each question

Additional Materials Needed

- Reference Sheet
- Multiple Choice Answer Sheet
- 6 Writing books

Structure & Suggested Time Spent

Section I
Multiple Choice Questions

- Answer Q1 – 10 on the multiple choice answer sheet
- Allow 18 minutes for this section

Section II
Extended response Questions

- Attempt all questions in this section in a separate writing booklet
- Allow 162 minutes for this section

This paper must not be removed from the examination room

Disclaimer
The content and format of this paper does not necessarily reflect the content and format of the HSC examination paper.

Section I

10 Marks

Allow about 18 minutes for this section

Use the multiple choice answer sheet for Questions 1 - 10.

Question 1

A polynomial has a double root at $z=i$. It has real coefficients and a leading coefficient of one.

Which of the following could be that polynomial?

(A) $P(z)=(z+i)^2(z-i)^2(z+4)$

(B) $P(z)=(z+i)^2(z+4)$

(C) $P(z)=(z+i)(z-i)^2$

(D) $P(z)=(z-i)^2$

Question 2

An ellipse with centre at the origin and whose major axis is twice as large as its minor axis has its foci at $x=\pm 3$.

Which of the following equations could represent this ellipse?

(A) $5x^2+20y^2=36$

(B) $5x^2-20y^2=36$

(C) $x^2+4y^2=12$

(D) $x^2-4y^2=12$

Question 3

When using integration by parts to solve the integral $\int 3x^3 \sin\left(x^2\right)\ dx$ which combination would be the best first step?

(A) $u = 3x^3$ and $\dfrac{dv}{dx} = \sin x^2$

(B) $u = 3x^2$ and $\dfrac{dv}{dx} = x\sin x^2$

(C) $u = 3x\sin x^2$ and $\dfrac{dv}{dx} = 3x^2$

(D) $u = \sin x^2$ and $\dfrac{dv}{dx} = 3x^3$

Question 4

If $z = \cos\theta + i\sin\theta$ then what is the value of $z + \dfrac{1}{z}$?

(A) $\cos 2\theta$

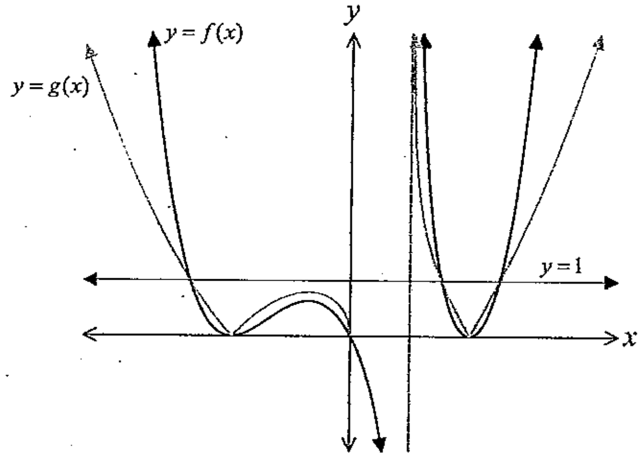
(B) $2\cos\theta$

(C) $i\sin 2\theta$

(D) $2i\sin\theta$

Question 5

Below is a sketch of $y = f(x)$ (Black) and $y = g(x)$ (Light Grey).



Which of the following is the correct relationship between the two functions?

(A) $g(x) = \left(f(x)\right)^2$

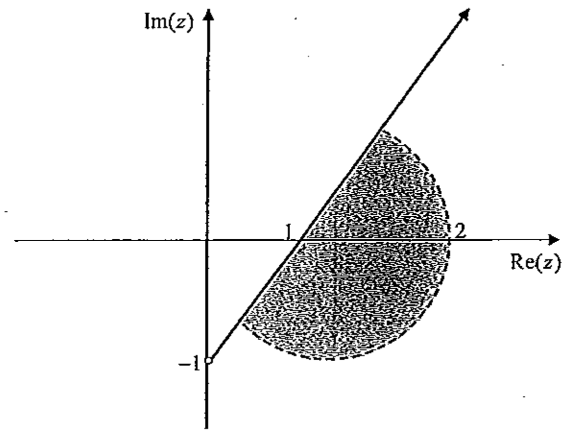
(B) $g(x) = \sqrt{f(x)}$

(C) $\left(g(x)\right)^2 = f(x)$

(D) $g(x) = \left(f(x)\right)^3$

Question 6

The sketch below is a region in the Argand plane.



Which combination describes the shaded region above?

(A) $|z-1|<1$ and $0 \le Arg(z+i) \le \frac{\pi}{4}$

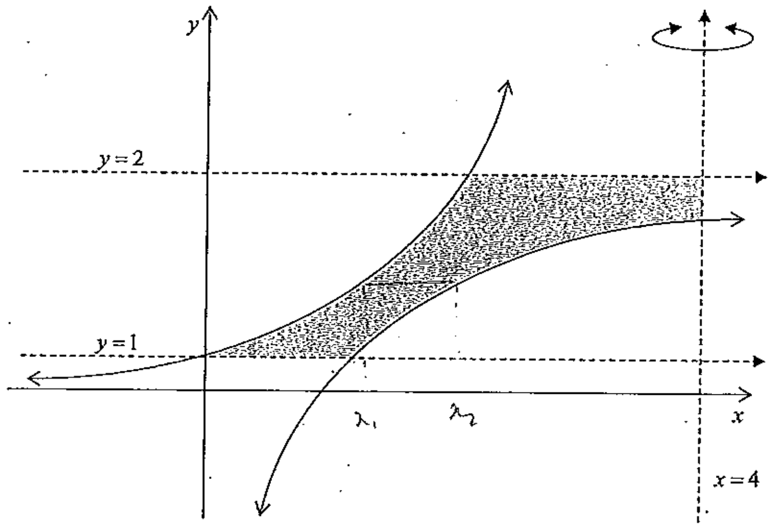
(B) $|z+1|<1$ and $0 \le Arg(z-i) \le \frac{\pi}{4}$

(C) $|z+1|<1$ and $0 \le Arg(z+i) \le \frac{\pi}{4}$

(D) $|z-1|<1$ and $0 \le Arg(z-i) \le \frac{\pi}{4}$

Question 7

The area between the curves $y=e^x$, $y=\ln x$ and the lines $y=1$, $y=2$ and $x=4$ is rotated around the line $x=4$.



Which of the following integrals will give the volume of the solid produced?

(A) $\pi\int_1^2 (4-\ln y)^2-(4-e^y)^2 \, dy$

(B) $\pi\int_1^2 (4-e^y)^2-(4-\ln y)^2 \, dy$

(C) $\pi\left[\int_1^{\ln 4} (4-\ln y)^2-(4-e^y)^2 \, dy+\int_{\ln 4}^2 (4-\ln y)^2 \, dy\right]$

(D) $\pi\left[\int_1^{\ln 4} (4-e^y)^2-(4-\ln y)^2 \, dy+\int_{\ln 4}^2 (4-e^y)^2 \, dy\right]$

Question 8

Which of the following is the equation of a normal drawn to a rectangular hyperbola at the point $P\left(cp, \frac{c}{p}\right)$?

(A) $p^3x + py + c(1-p^4) = 0$

(B) $p^3x - py + c(1-p^4) = 0$

(C) $p^3x + py + c(p^4-1) = 0$

(D) $p^3x - py + c(p^4-1) = 0$

Question 9

Using $t = \tan\frac{\theta}{2}$, which of the following is equivalent to $\int \frac{4\sin\theta}{2-\cos\theta}\, d\theta$?

(A) $\int \frac{16t}{(t^2+1)(1+3t^2)}\, dt$

(B) $\int \frac{8}{(3t^2+1)}\, dt$

(C) $\int \frac{1-t^2}{(t^2+1)(t^2-t+1)}\, dt$

(D) $\int \frac{1-t^2}{2(t^2-t+1)}\, dt$

Question 10

Which of the following integrals is always equal to $\int_b^a f(x)\, dx$?

(A) $\int_0^a f(x)\, dx + \int_0^b f(x)\, dx$

(B) $\int_0^a f(-x) + f(x)\, dx + \int_b^{-a} f(x)\, dx$

(C) $\int_{-b}^{-a} f(-x)\, dx$

(D) $\int_0^{-a} f(-x)\, dx + \int_0^{-b} f(x)\, dx$

END OF SECTION I

Section II

90 Marks

Allow about 162 minutes for this section

Answer questions 11 - 16 in separate booklets.

Question 11 **Begin a new booklet** **15 Marks**

a) Given $z=3+i$ and $w=4-i\sqrt{3}$ find $\frac{z}{\overline{w}}$ showing all working. **1**

b) k is a complex number in the first quadrant with argument α.

Sketch k, ik and k^2 on an Argand plane showing the angles between them. **2**

c) Solve $z^2-(7-i)z+14-5i=0$. **3**

d) A variable point z moves in the Argand plane so that $\sqrt{2}|z-4-i|=\sqrt{3}|z-3+i|$.

Find the Cartesian representation of this locus and sketch it showing all important features. **3**

Question 11 continues on the next page.

e) Given $\dfrac{3x^3-x^2-6x-4}{(x^2+1)(x-3)}=A+\dfrac{Bx+C}{x^2+1}+\dfrac{D}{x-3}$

(i) Find the values of A, B, C and D. **2**

(ii) Hence or otherwise, find $\displaystyle\int\frac{3x^3-x^2-6x-4}{(x^2+1)(x-3)}\,dx$. **2**

f) Find $\int x\sin kx\,dx$. **2**

Question 12 **Begin a new booklet** **16 Marks**

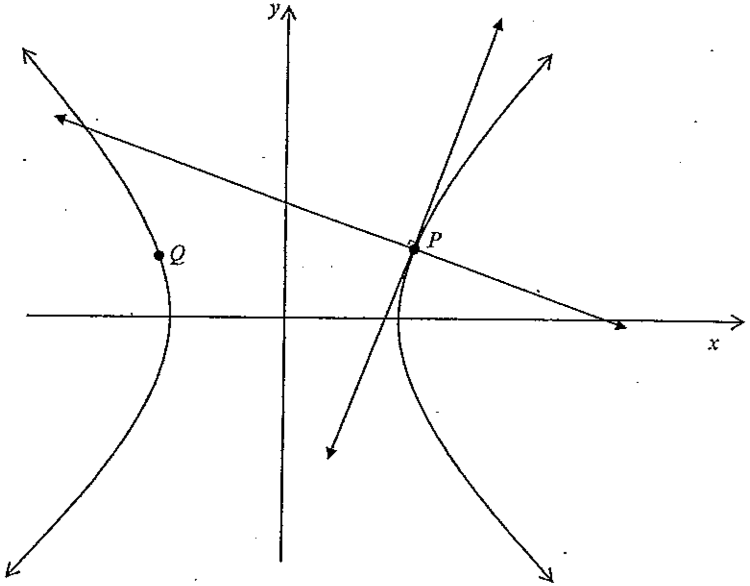
a) Factorise $P(z)=4z^3-24z^2+46z-30$ over the complex field. **3**

b) Given $T_1=0$, $T_2=9$ and $T_n=6T_{n-1}-9T_{n-2}$ for $n\geq 3$, use Mathematical induction to show that $T_n=(n-1)3^n$ for $n\geq 1$. **4**

c) Sketch $4x^2-y^2=20$ showing all intercepts, asymptotes, directricies and foci. **2**

Question 12 continues on the next page.

d) The Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $a>b$ has a variable point $P(a\sec\theta, b\tan\theta)$ on its right hand branch. A point Q lies on the left hand branch. The normal at Q meets the normal at P at the point N.



(i) Show that the equation of the normal at P is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$. **2**

(ii) Show that when Q is $\left(a\operatorname{cosec}\theta, \frac{b}{\tan\theta}\right)$ then the x coordinate of N is

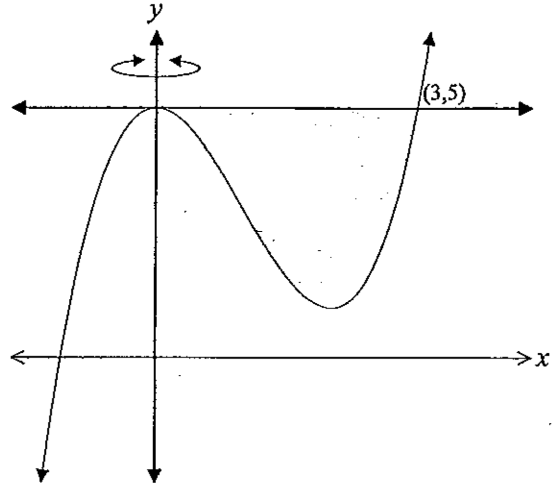
$\frac{(1+\cot\theta)(a^2+b^2)}{a\cos\theta}$. **3**

e) Given $m>0$ and $n>0$ show that $\frac{m^2}{n^2}+\frac{n^2}{m^2}\geq 2$. **2**

END OF QUESTION 12

Question 13 **Begin a new booklet** **15 Marks**

a) The area bound by the curve $y=x^3-3x^2+5$ and $y=5$ is rotated around the y-axis.



Using the method of cylindrical shells, find the volume of the solid produced. **4**

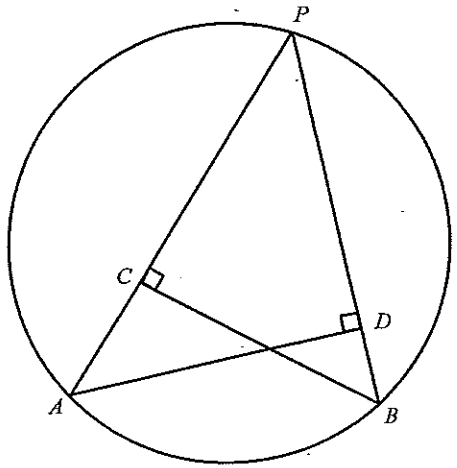
b) Integrate $\int \cos 2x \sin^3 x \, dx$. **3**

c) Integrate $\int \frac{\sqrt{1+x^2}}{x^4}\,dx$ using the substitution $x=\tan\theta$. **3**

Question 13 continues on the next page.

d) 3 points, A, B, and P lie on a circle. The points C and D lie on AP and BP such that

$\angle ACB = \angle ADB = 90^\circ$.



(i) Show that $ACDB$ is a cyclic quadrilateral. **1**

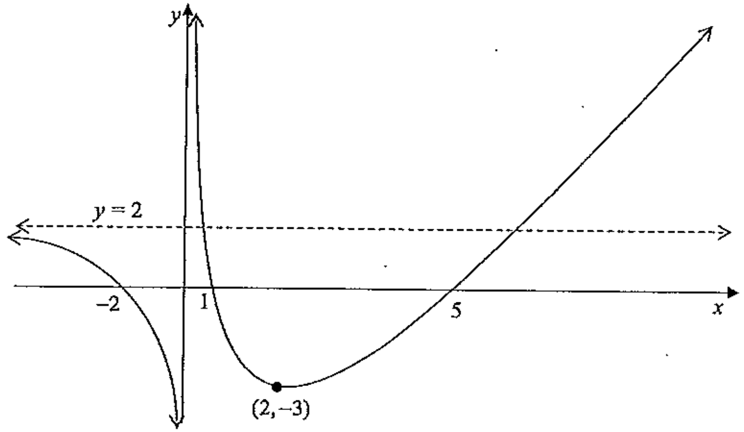
(ii) Show that ΔCPD and ΔAPB are similar. **2**

(iii) Use the previous parts or otherwise to show that as P moves around the circle the length CD remains constant. **2**

END OF QUESTION 13

Question 14 **Begin a new booklet** **15 Marks**

a) Below is a sketch of $y = f(x)$. It has x intercepts at -2, 1 and 5 as well as a local minimum at $(2,-3)$.



Sketch the following showing any asymptotes, limits to infinity, intercepts and turning points.

(i) $y = \dfrac{1}{f(x)}$ **3**

(ii) $y = \left(f(x)\right)^2$ **2**

Question 14 continues on the next page.

b) The polynomial $P(z)=rz^4-\frac{sz^3}{3}+\left(4r^2-s^2\right)z+r-s$ has a double root at $z=\frac{r}{s}$ where both r and s are real positive numbers.

(i) Show that $\frac{4r^4}{s^3}-\frac{r^2}{s}+4r^2-s^2=0$ **1**

(ii) Hence show that $s=2r$. **1**

(iii) Using part (ii) find the value of r if the sum of the squares of the other two roots is zero. **2**

c) The sum of a complex number and its conjugate is twice the real component.

Using this and a binomial expansion it can be shown that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

(i) Using the above statement find all six roots of the equation

$64x^6-112x^4+56x^2-7=0$. **1**

(ii) Hence show that $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}=\frac{\sqrt{7}}{8}$. **2**

d) Integrate: $\int \frac{4x}{\sqrt{2x-3x^2}}\,dx$. **3**

END OF QUESTION 14

Question 15 **Begin a new booklet** **14 Marks**

a) Let $\omega=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$.

(i) Show that $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$. **2**

(ii) The cubic $x^3+ax^2+bx+c=0$ has roots $\omega+\omega^6$, $\omega^2+\omega^5$ and $\omega^3+\omega^4$.

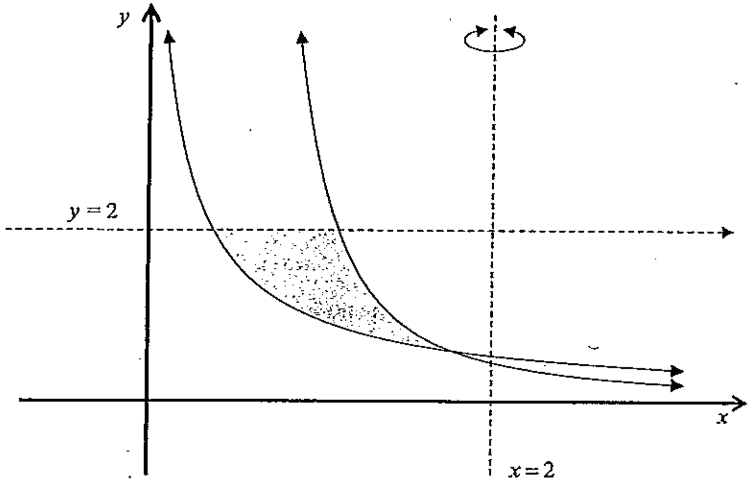
Show that $a=1$, $b=-2$ and $c=-1$. **2**

(iii) Hence find the value of $\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{\pi}{7}$. **3**

Question 15 continues on the next page.

b) The curves $y=\frac{1}{x}$ and $y=\frac{4}{(2x-1)^2}$ intersect in the first quadrant at the point $\left(\frac{2+\sqrt{3}}{2},2\left(2-\sqrt{3}\right)\right)$. You do not need to show this.

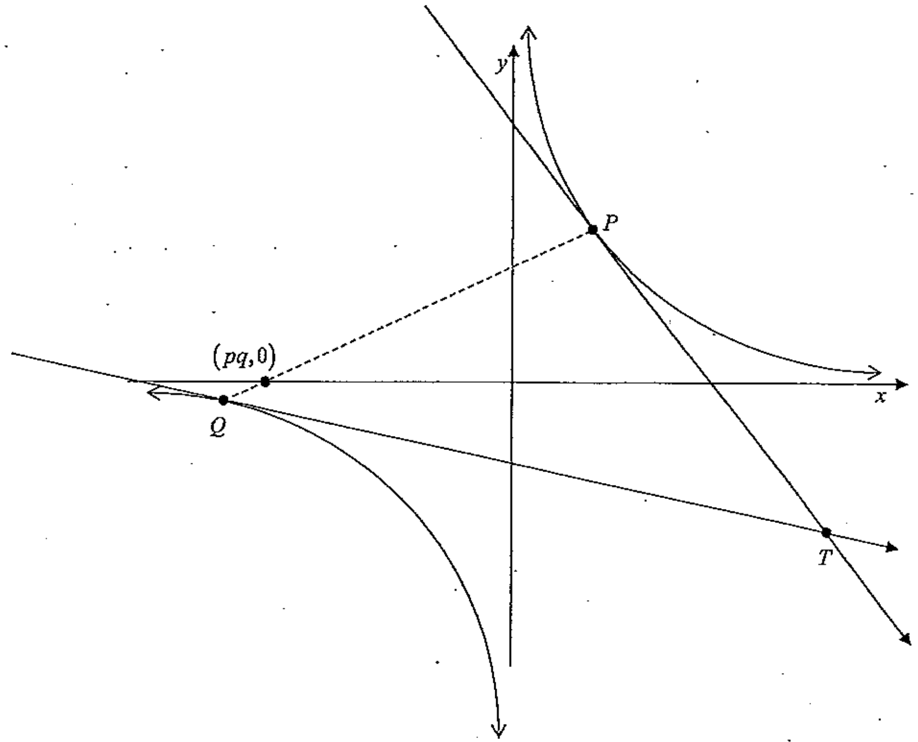
The area enclosed by the line $y=2$ and the curves $y=\frac{1}{x}$ and $y=\frac{4}{(2x-1)^2}$ is rotated around the line $x=2$.



Find the volume of the solid formed using slices that are perpendicular to the axis of rotation. Give you answer to 2 decimal places. **4**

Question 15 continues on the next page.

c) Two tangents are drawn to the rectangular hyperbola $xy=c^2$, one at $P\left(cp,\frac{c}{p}\right)$ which is on the right hand branch and one at $Q\left(cq,\frac{c}{q}\right)$ which is on the left hand branch. The equation of the tangent at P is $x+p^2y-2cp=0$.



(i) Show that the equation of the chord PQ is: $\frac{x}{pq}+y=c\left(\frac{p+q}{pq}\right)$ **1**

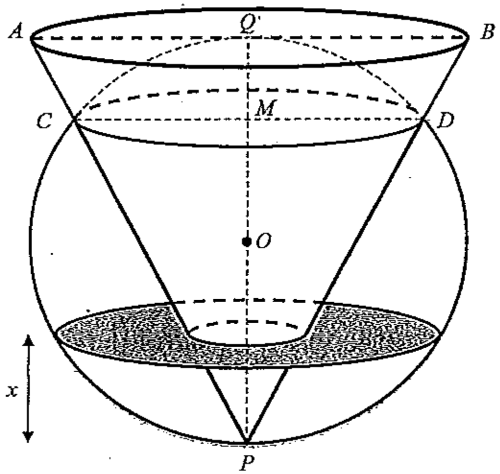
(ii) Find the equation of the locus of T, the point of intersection of the tangents given that the chord PQ always goes through $(pq,0)$. **2**

END OF QUESTION 15

Question 16 **Begin a new booklet** **15 Marks.**

a) A solid sphere, of radius R, has an inverted cone object drilled into it. The cone has a base radius R and a height $2R$. P, the apex of the cone, is directly below O, the centre of the sphere.

The cone's base has diameter AB and centre Q. PQ runs through the centre of the sphere and M is the centre of CD. A vertical plane through $ABCD$ passes through O and M.



A slice is taken at a perpendicular height x from the point P.

(i) Using similar triangles or otherwise, show that the length OM is $\frac{3}{5}R$. **2**

(ii) Show that the volume of the shaded slice is given by $\pi\left(2Rx-\frac{5x^2}{4}\right)\Delta x$. **3**

Question 16 continues on the next page.

b) Given $I_n=\int_0^{2\pi}(1+\cos x)^n\ dx$.

Show that $I_{n+1}=\frac{2n+1}{n+1}I_n$ for $n\geq 1$. **4**

c) Consider the sketch $y=\ln x$.

(i) Show that $y=\ln x$ is concave down for all real x. **1**

(ii) Use the trapezoidal rule to estimate the area bound by the curve, $x=1$ and $x=n$ using trapezia of width 1 unit. **1**

(iii) Show that the exact area bound by the curve, the x-axis, the lines $x=1$ and $x=n$ is equal to $1-n+n\ln(n)$. **2**

(iv) Hence show that $n!<\frac{en^{n+\frac{1}{2}}}{e^n}$. **2**

END OF EXAM

Extension 2 Trial 2016

Solutions

Multiple Choice

Q1 As it has a double root at α it must also have a double root at negative α (as all coefficients are real)

Must be A

A

Q2 $2b = a$ & $ae = 3$

$b^2 = a^2(1 - e^2)$ As this is an ellipse

$\frac{a^2}{4} = \cancel{a^2}\left(1 - \frac{9}{\cancel{a^2}}\right)$

$a^2 = 4(a^2 - 9)$

$a^2 = 4a^2 - 36$

$3a^2 = 36$

$a^2 = 12$

$a = 2\sqrt{3}$

$b = \sqrt{3}$

$\frac{x^2}{12} + \frac{y^2}{3} = 1$

$x^2 + 4y^2 = 12$

C

Q3 B

This will leave us with an integral we can do

$(x \sin x^2)$

Q4 $\cos\theta + i\sin\theta$

$\frac{1}{z} = \bar{z}$ As the modulus is one

$z + \frac{1}{z} = 2\cos\theta$ B

Q5 - Values of $f(x)$ that are less than one increase
- Greater than one decrease
- No negative of $f(x)$ are considered
- $f(x) = g(x)$ when $f(x) = 1$
- Not reflected about the x axis

B

Q6 Centre of circle is $(1,0)$

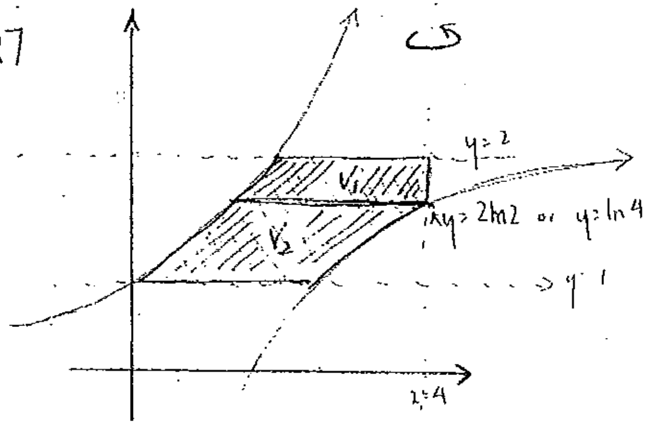
$|z-1| < 1$

Argument taken from $(0,-1)$

$\therefore \quad 0 \le \operatorname{Arg}(z+i) \le \frac{\pi}{4}$

$\therefore$ (A)

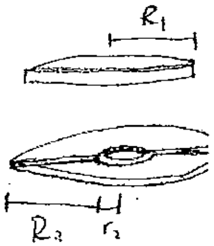
Q7



To find this volume you need to find two seperate volumes

V_1 - Defined by $y=e^x$ only

V_2 - Defined by $y=e^x$ & $y=\ln x$



$$V_T = \pi\int_{\ln 4}^{2} (R_1)^2\, dy + \pi\int_{1}^{\ln 4} (R_2)^2 - (r_2)^2\, dy$$

$$= \pi\int_{\ln 4}^{2} (4-\ln y)^2\, dy + \pi\int_{1}^{\ln 4} (4-\ln y)^2 - (4-e^y)^2\, dy$$

$\therefore$ (C)

Q8 $xy = c^2$

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

@P $\quad \frac{dy}{dx} = -\frac{1}{p^2}$

$$m_N = p^2$$

$$\therefore \quad p^2 = \frac{y - \frac{c}{p}}{x - cp}$$

$$p^2x - cp^3 = y - \frac{c}{p}$$

$$p^3x - cp^4 - py + c = 0$$

$$p^3x - py + c(1-p^4) = 0 \quad (B)$$

Q9 $\displaystyle\int \frac{4\sin\theta}{2-\cos\theta}\, d\theta \qquad t = \tan\frac{\theta}{2}$

then $\quad d\theta = \dfrac{2dt}{1+t^2}$

$$\therefore I = \int \frac{4\left(\frac{2t}{1+t^2}\right)}{2-\left(\frac{1-t^2}{1+t^2}\right)}\, \frac{2dt}{1+t^2}$$

$$= \int \frac{16t}{(1+t^2)(1+3t^2)}\, dt \quad (A)$$

If $\int f(x)\,dx = F(x)$

Q10 $\int_b^a f(x)\,dx \quad F(a) - F(b)$

(A) $\int_0^a f(x)\,dx + \int_0^b f(x)\,dx$

$= F(a) - F(0) + F(b) - F(0)$

$= F(a) + F(b) - 2F(0)$

$\therefore$ Not $= \int_b^a f(x)\,dx$

(B) $\int_0^a f(-x) + f(x)\,dx + \int_b^{-a} f(x)\,dx$

$= \Big[-F(-x) + F(x)\Big]_0^a + \Big[F(x)\Big]_b^{-a}$

$= \Big[-F(-a) + F(a)\Big] - \Big[\cancel{-F(0)} + \cancel{F(0)}\Big]$

$+ \cancel{F(-a)} - F(b)$

$= F(a) - F(b) = \int_b^a f(x)\,dx$

$\therefore$ B

(C) $\int_{-b}^{-a} f(-x)\,dx$

$= \Big[-F(-x)\Big]_{-b}^{-a}$

$= -\big(F(a) - F(b)\big)$

$= F(b) - F(a) \neq \int_b^a f(x)\,dx$

(D) $\int_0^{-a} f(-x)\,dx + \int_0^{-b} f(x)\,dx$

$= \Big[-F(-x)\Big]_0^{-a} + \Big[F(x)\Big]_0^{-b}$

$= -F(a) + \cancel{F(0)} + F(-b) - \cancel{F(0)}$

$= -F(a) + F(-b) \neq \int_b^a f(x)\,dx$

$\therefore$ B

Question 11

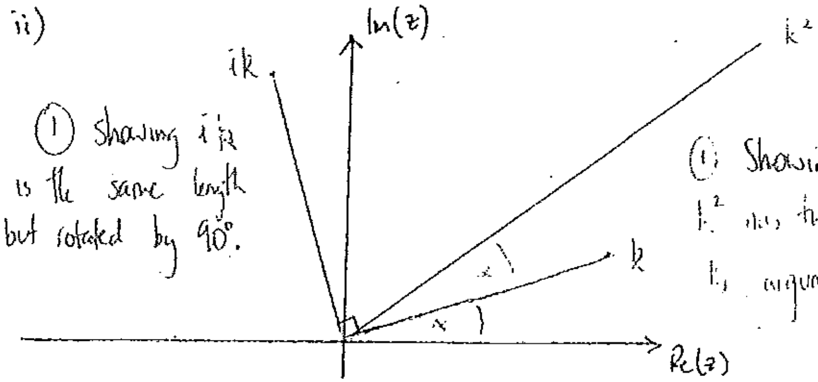
(a) i) $\dfrac{z}{w} = \dfrac{3+i}{4+i\sqrt{3}} \times \dfrac{4-i\sqrt{3}}{4-i\sqrt{3}}$

$= \dfrac{12 - 3\sqrt{3}i + 4i + \sqrt{3}}{16+3}$

$= \dfrac{(12+\sqrt{3}) + (4-3\sqrt{3})i}{19}$

Showing all working ①

ii)



① Showing ik is the same length but rotated by 90°.

① Showing k^2 has twice k's argument

let $Arg(z) = \alpha$

b) $z^2 - (7-i)z + 14 - 5i = 0$

$\Delta = [-(7-i)]^2 - 4 \times 1 \times (14-5i)$

$= 49 - 14i - 1 - 56 + 20i$

$= -8 + 6i$ ① Establishing the discriminant.

$z = \dfrac{7 - i \pm \sqrt{\Delta}}{2}$

let $\sqrt{-8+6i} = x + iy$

$x^2 - y^2 + 2xyi = -8 + 6i$

$x^2 - y^2 = -8$ ①

$2xy = 6$ ②

Comparing real & imaginary components.

from ② $y = \dfrac{6}{2x}$ ③

③ into ①

$x^2 - \left(\dfrac{3}{x}\right)^2 = -8$

$x^4 - 9 = -8x^2$

$x^4 + 8x^2 - 9 = 0$

$(x^2+9)(x^2-1) = 0$ ① Find the square root of Δ

As x is real $x = \pm 1$ $y = \pm 3$

$\sqrt{-8+6i} = \pm(1+3i)$

$$z = \frac{7-i \pm (1+3i)}{2}$$

$z = 4+i$ & $3-2i$ (1) Correct solutions.

(c) $\sqrt{2}\,|z-(4+i)| = \sqrt{3}\,|z-(3-i)|$

Not a standard locus

let $z = x+iy$

$$\sqrt{2}\,|(x-4)+i(y-1)| = \sqrt{3}\,|(x-3)+i(y+1)|$$

(1) Using cartesian equivalent

$$2\left[(x-4)^2+(y-1)^2\right] = 3\left[(x-3)^2+(y+1)^2\right]$$

$$2\left(x^2-8x+16+y^2-2y+1\right) = 3\left(x^2-6x+9+y^2+2y+1\right)$$

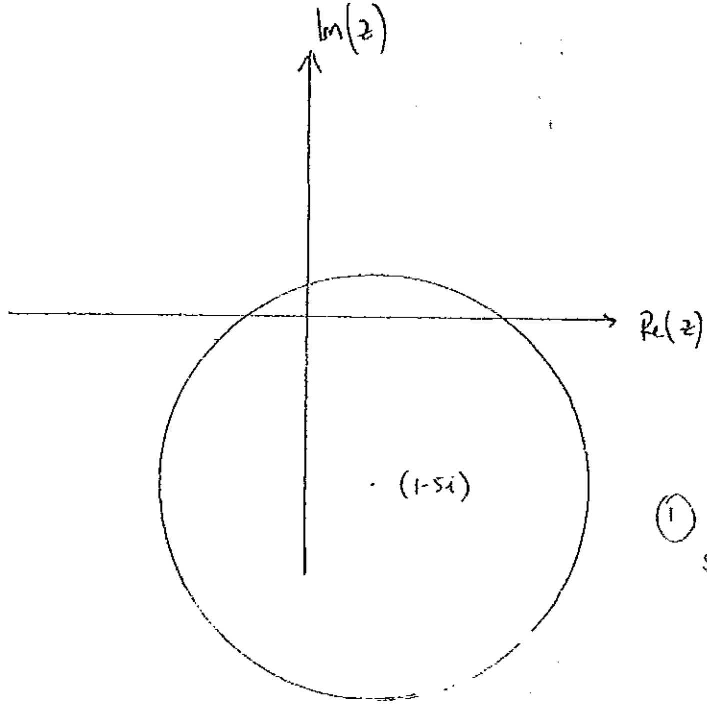
$$\cancel{2x^2}-\cancel{16x}+32+\cancel{2y^2}-\cancel{4y}+2 = \cancel{3x^2}-\cancel{18x}+27+3y^2+6y+$$

$$26+4 = x^2-2x+1+y^2+10y+25$$

$$(x-1)^2+(y+5)^2 = 30$$

Circle ; centre $(1,-5)$, radius of $\sqrt{30}$

(1) finding the cartesian equivalent.



(1) Correct sketch.

d) $$\frac{3x^3-x^2-6x-4}{(x^2+1)(x-3)} = A + \frac{Bx+C}{x^2+1} + \frac{D}{x-3}$$

$A = 3$ (leading coefficient)

$$3x^3-x^2-6x-4 = 3(x^2+1)(x-3) + (Bx+C)(x-3) + D(x^2$$

When $x=3$

$$50 = 10D$$

$$D = 5$$

When $x=0$

$-4 = -9 - 3C + 5$

$C = 0$

(1) Working towards the right answer

When $x=1$

$-8 = -12 + (-2)B + 10$

$-6 = -2B$

$B = 3$

(1) All values correct.

$$\frac{3x^3 - x^2 - 6x - 4}{(x^2+1)(x-3)} = 3 + \frac{3x}{x^2+1} + \frac{5}{x-3}$$

ii) $\therefore \; I = \int 3 + \frac{3x}{x^2+1} + \frac{5}{x-3} \, dx$

$$= 3x + \frac{3}{2}\ln(x^2+1) + 5\ln(x-3)$$

(1) $+C$ (1)

e) $\int x \sin kx \, dx$

$u = x$ $\qquad v = \frac{-\cos kx}{k}$

$\frac{du}{dx} = 1$ $\qquad \frac{dv}{dx} = \sin kx$

(1) Using integration by parts

$$I = -\frac{x\cos kx}{k} + \int \frac{\cos kx}{k} \, dx$$

$$= \frac{-x\cos kx}{k} + \frac{\sin kx}{k^2} + C$$

(1) final answer.

Question 12

a) $P(z) = 4z^3 - 24z^2 + 46z - 30$

$P(3) = 0$ (1)

$\therefore P(z) = (z-3)(az^2 + bz + c)$

$a = 4$, & $c = 10$ (by inspection)

from z^2

$-3a + b = -24$

$b = -12$

$\therefore P(z) = (z-3)(4z^2 - 12z + 10)$

$= 2(z-3)(2z^2 - 6z + 5)$ (1)

let $2z^2 - 6z + 5 = 0$

$z = \dfrac{6 \pm \sqrt{36 - 4\times2\times5}}{4}$

$= \dfrac{6 \pm \sqrt{-4}}{4}$

$= \dfrac{6 \pm 2i}{4}$

$= \underline{\dfrac{3 \pm i}{2}}$

$\therefore P(z) = 4(z-3)\left(z - \left(\frac{3+i}{2}\right)\right)\left(z - \left(\frac{3-i}{2}\right)\right)$

(1)

factorised form

not

(b) $T_1 = 0$ $T_2 = 9$ $T_n = 6T_{n-1} - 9T_{n-2}$

Show that $T_n = (n-1)3^n$

(1) Show true for $n = 3$

$T_3 = 6\times - 9\times$

$= 54$ (1)

$T_3 = (3-1)\times 3^3$

$= 54$ $\therefore$ True for $n = 3$

(2) Assume true for $n = k$ & $k-1$

$T_k = (k-1)3^k$ $T_{k-1} = (k-2)3^{k-1}$ (1)

(3) Prove true for $n = k+1$ [Prove $T_{k+1} = (k)3^{k+1}$]

$T_{k+1} = 6T_k - 9T_{k-1}$

$= 6\left[(k-1)3^k\right] - 9\left[(k-2)3^{k-1}\right]$

$= 3.3^{k-1}\left[2(k-1).3 - 3(k-2)\right]$

$= 3^k\left[6k - 6 - 3k + 6\right]$

$= 3^k[3k] = k[3^{k+1}]$

$\therefore$ Proven by Mathematical Induction

(c) $4x^2 - y^2 = 20$ $\quad\therefore\ a = \sqrt{5}\ ,\ b = 2\sqrt{5}$

$$\frac{x^2}{5} - \frac{y^2}{20} = 1$$

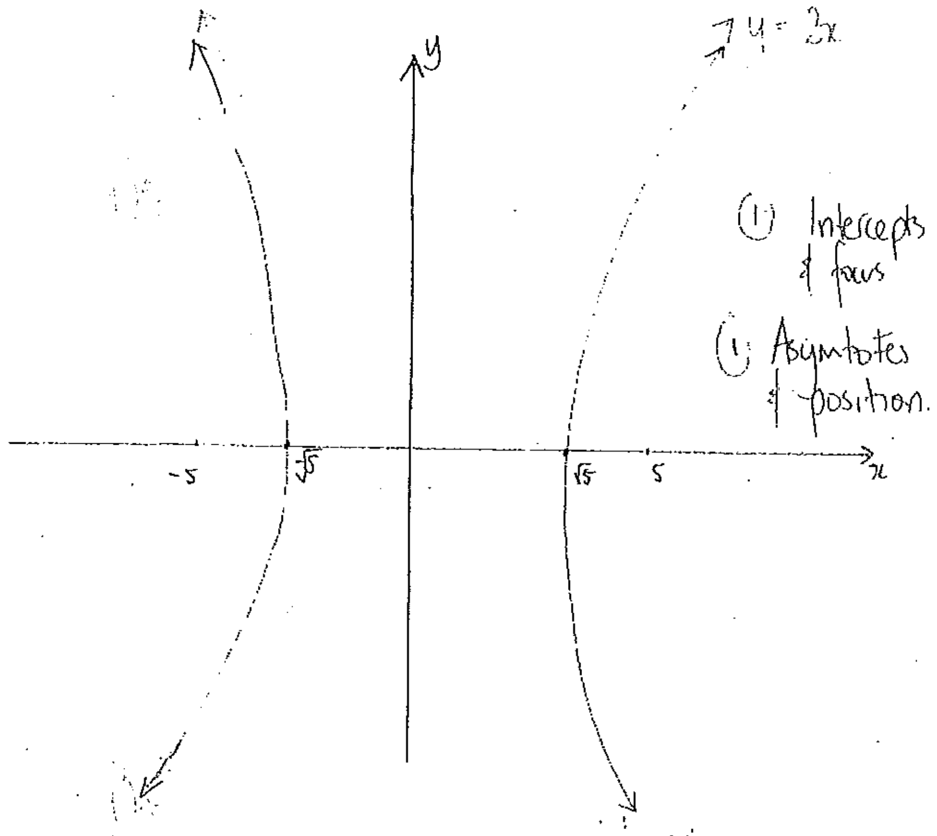
$ae = 5$

$$b^2 = a^2(e^2-1)$$

$$20 = 5(e^2-1)$$

$$4 = e^2 - 1$$

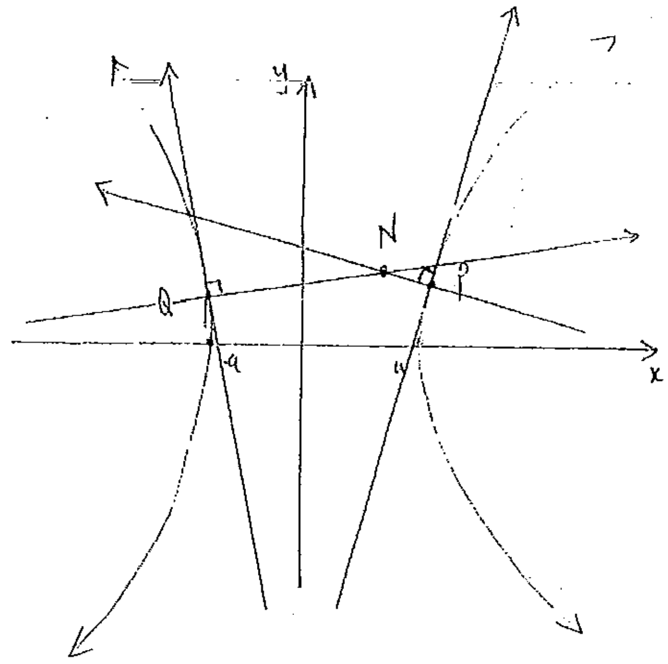
$$e = \sqrt{5}$$



① Intercepts & focus

① Asymtotes & position.

(d)



(i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$\frac{\cancel{2}x}{a^2} - \frac{\cancel{2}y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{xb^2}{ya^2}$$

let P have coordinates (x_P, y_P)

At P $\quad M_N = \dfrac{-y_P a^2}{x_P b^2}$ ① With all steps before.

Equation of Normal

$$-\frac{y_p a^2}{x_p b^2} = \frac{y - y_p}{x - x_p}$$

$$\times \frac{b^2}{y_p}$$

$$-\frac{a^2}{x_p}(x - x_p) = \frac{b^2}{y_p}(y - y_p)$$

$$-\frac{a^2}{x_p}x + a^2 = \frac{b^2 y}{y_p} - b^2$$

$$\frac{a^2 x}{x_p} + \frac{b^2 y}{y_p} = a^2 + b^2$$

$x_p = a\sec\theta$ & $y_p = b\tan\theta$

$$\therefore \quad \frac{a^2 x}{a\sec\theta} + \frac{b^2 y}{b\tan\theta} = a^2 + b^2$$

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$

As required. (1) Showing steps.

ii) If Q lies on the hyperbola with y coordinate of $\frac{b}{\tan\theta}$ the the x coordinate is

$a\operatorname{cosec}\theta$

(Sub $b\cot\theta$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$)

$\therefore$ Equation of Normal at Q

$$\frac{ax}{\operatorname{cosec}\theta} + \frac{by}{\cot\theta} = a^2 + b^2$$

@P $$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$

Solve simultaneously to find N

from (1) $$y = \frac{\cot\theta}{b}\left[a^2 - b^2 - \frac{ax}{\operatorname{cosec}\theta}\right]$$

(1) Solving simoltaneously

sub (3) into (2)

$$\frac{ax}{\sec\theta} + \frac{\cancel{b}}{\tan\theta} \times \frac{\cot\theta}{\cancel{b}}\left[a^2+b^2-\frac{ax}{\operatorname{cosec}\theta}\right] = a^2+b^2$$

$$\frac{ax}{\sec\theta} + \frac{1}{\tan^2\theta}(a^2+b^2) - \frac{1}{\tan^2\theta}\cdot\frac{ax}{\operatorname{cosec}\theta} = (a^2+b^2)$$

$$(a\cos\theta)x - (a\cos\theta\cot\theta)x = a^2+b^2-\cot^2\theta(a^2+b^2)$$

$$\left[a\cos\theta(1-\cot\theta)\right]x = (1-\cot^2\theta)\left[a^2+b^2\right]$$

$$x = \frac{\cancel{(1-\cot\theta)}(1+\cot\theta)(a^2+b^2)}{a\cos\theta\,\cancel{(1-\cot\theta)}}$$

$$= \frac{(1+\cot\theta)(a^2+b^2)}{a\cos\theta}$$ As required.

(1) Arriving at the desired result.

(e) $a>0$, $b>0$

$$(a-b)^2 \geq 0$$

$$a^2-2ab+b^2 \geq 0$$

+ 4ab both sides

$$a^2+2ab+b^2 \geq 4ab$$

$$(a+b)^2 \geq 4ab$$

$$a+b \geq 2\sqrt{ab}$$

(1) Using a correct inequality

let $a=\left(\frac{a}{b}\right)^2$, $b=\left(\frac{b}{a}\right)^2$

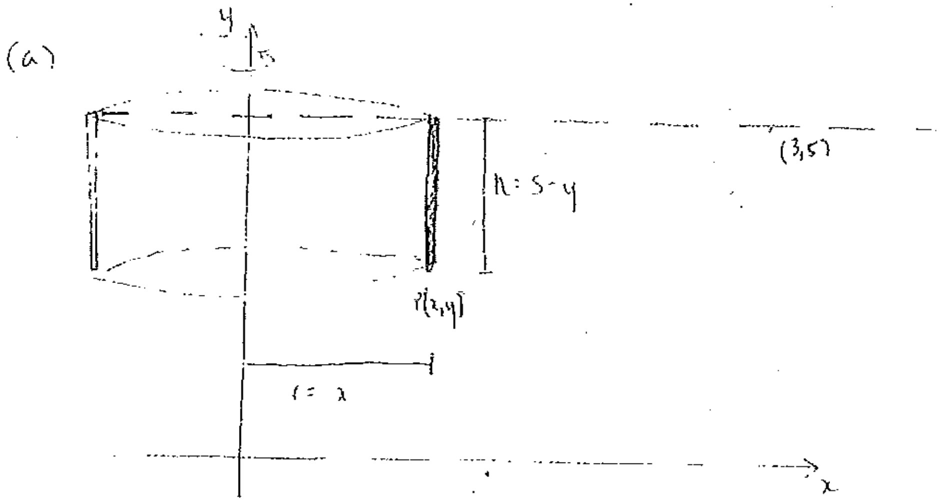
$$\frac{a^2}{b^2}+\frac{b^2}{a^2} \geq 2\sqrt{\frac{a^2}{b^2}\times\frac{b^2}{a^2}}$$

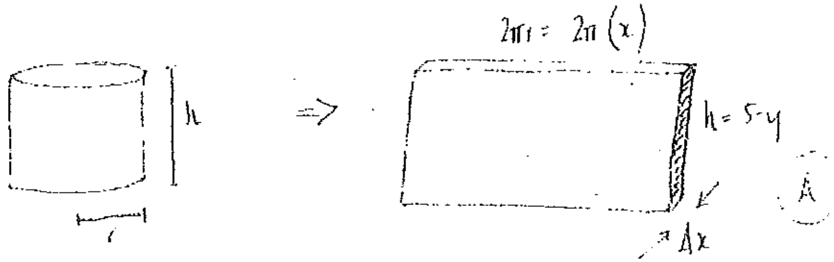
$$\frac{a^2}{b^2}+\frac{b^2}{a^2} \geq 2$$ As required.

(1) Worked through to the desired result

Question 13

(a)





$\Delta V = 2\pi r h \, . \, \Delta x$

$= 2\pi x \, (5 - y) \, \Delta x \qquad y = x^3 - 3x^2 + 5$

$\Delta V = 2\pi x \left(5 - x^3 + 3x^2 - 5 \right) \Delta x$

$= 2\pi x \left(-x^3 + 3x^2 \right) \Delta x$

$\Delta V = 2\pi \left(-x^4 + 3x^3 \right) \Delta x$ (B)

$V \approx \sum_{x=0}^{3} 2\pi \left(-x^4 + 3x^3 \right) \Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{3} 2\pi \left(-x^4 + 3x^3 \right) \Delta x$ (C)

$= 2\pi \int_0^3 -x^4 + 3x^3 \, dx$

$= 2\pi \left[-\frac{x^5}{5} + \frac{3x^4}{4} \right]_0^3$

$= \frac{27\pi}{2}$ units^3 (D)

Marking Outline

1. Correct dimensions of typical slice (A)
1. Changing the integral to be in terms of one variable (B)
1. Turning of a typical slice into a sum of infinitly small slices (ie, an integral) (C)
1. Correct answer. (D)

(b) $\int \cos 2x \sin^3 x \, dx$

$= \int (2\cos^2 x - 1) \sin^3 x \, dx$ (1)

$= \int (2\cos^2 x - 1)(1-\cos^2 x) \sin x \, dx$

let $u = \cos x$ $\quad \frac{du}{dx} = -\sin x$

$\therefore I = -\int (2u^2-1)(1-u^2) \, du$ (1) Correct substitution.

$= -\int 2u^2 - 2u^4 - 1 + u^2 \, du$

$= -\int -2u^4 + 3u^2 - 1 \, du$

$= -\left[-\frac{2u^5}{5} + u^3 - u\right] + c$

$= \cos x - \cos^3 x + \frac{2\cos^5 x}{5} + c$

(1) In terms of x.

(c) $\int \frac{\sqrt{1+x^2}}{x^4} \, dx$ $\quad x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$

$\therefore I = \int \frac{\sqrt{1+\tan^2\theta}}{\tan^4\theta} \, d\theta \cdot \sec^2\theta$ (1)

$= \int \frac{\sec\theta}{\tan^4\theta} \, d\theta \cdot \sec^2\theta$

$= \int \frac{\sec^3\theta}{\tan^4\theta} \, d\theta$

$= \int \frac{1}{\cancel{\cos^3\theta}} \cdot \frac{\cos^4\theta}{\sin^4\theta} \, d\theta$

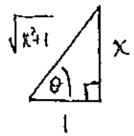
$= \int \frac{\cos\theta}{\sin^4\theta} \, d\theta$

$= \int (\sin\theta)^{-4} \cdot \left(\frac{d}{d\theta}\sin\theta\right) d\theta$

$= \frac{(\sin\theta)^{-3}}{-3} + c$ (1)

$$= -\frac{1}{3\sin^3\theta} + C$$

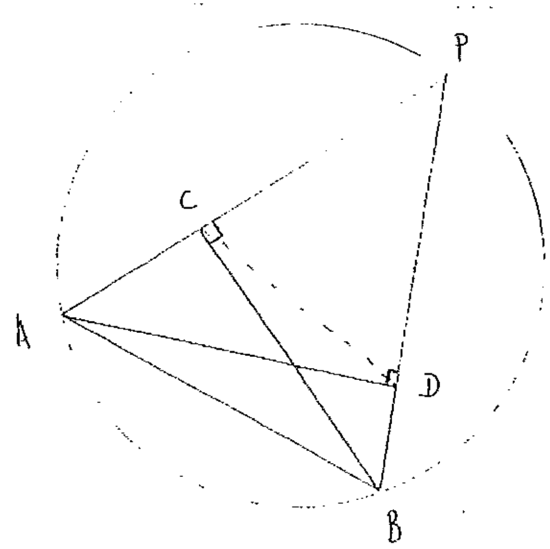
If $x = \tan\theta$



$$= \frac{-1}{3} \times \left(\frac{\sqrt{x^2+1}}{x}\right)^3 + C$$

$$= \frac{-\sqrt{(x^2+1)^3}}{3x^3} + C$$ (1) In terms of x

(d)



(i) $\angle ACB = \angle ADB = 90^\circ$ given

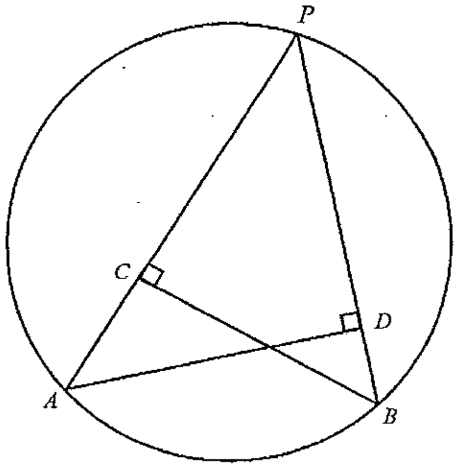
ACDB is a cyclic quadrilateral as the chord AB subtends the same angle at both C & D (Chords subtend equal angles at the circumference) (1)

ii) Prove $\triangle CPD \; ||| \; \triangle APB$

$\angle P$ is common

$\angle PDC = \angle PAB$ (external $\angle$ of cyclic quadrilateral (ACDB) is equal to the interior opposite) (1)

$\triangle CPD \; ||| \; \triangle APB$ (equiangular) (1) iii)



let $\angle APB = \alpha$

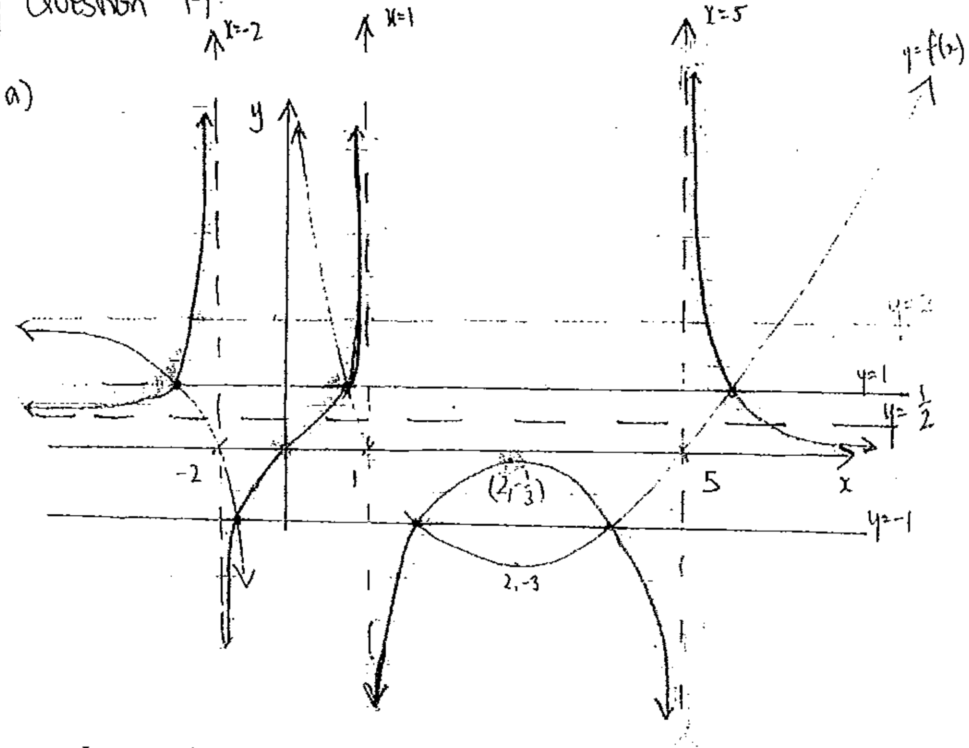
α is a constant as the chord AB subtends the same angle regardless of P's position (assuming P remains in the major segment.) (1)

In ΔPCB, $\angle PBC = 90 - \alpha$ ($\angle$ sum of Δ)

$\therefore$ In circle ACDB, chord CD subtends $\angle DBC$, which equal $90 - \alpha$
$90 - \alpha$ is a fixed value (as α is constant) (1)
$\therefore$ CD must be constant length as

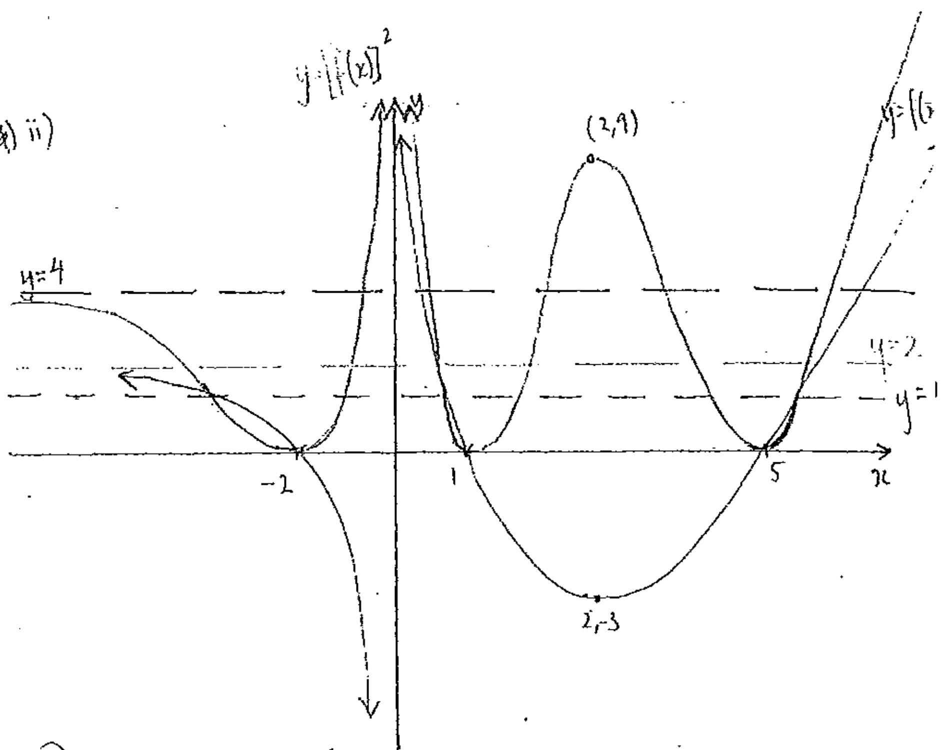
Question 14.

a)



(1) Limits - All limit correct
(1) Turning Pt - labeled correctly
(1) Asymptotes - All zeros are asymptotes & 0,0 excluded.

(a) ii)



(1) limits to infinity

Turning pt @ (2,9) [Maximum]

Turning pts @ zeros (minimums)

(b) i) $P(z) = rz^4 - \frac{sz^3}{3} + (4r^2 - s^2)z + r - s$

double root at $z = \frac{r}{s}$

$P'(z) = 4rz^3 - sz^2 + 4r^2 - s^2$

$P'\left(\frac{r}{s}\right) = 0$ As $\frac{r}{s}$ is a double root (1)

$$4r\left(\frac{r}{s}\right)^3 - s\left(\frac{r}{s}\right)^2 + 4r^2 - s^2 = 0$$

$$\frac{4r^4}{s^3} - \frac{r^2}{s} + 4r^2 - s^2 = 0$$

ii) $4r^4 - r^2s^2 + 4r^2s^3 - s^5$

$$r^2(4r^2 - s^2) + s^3(4r^2 - s^2) = 0$$

$$(r^2 + s^3)(4r^2 - s^2) = 0$$

Must explain why not a solution. $\Rightarrow r^2 \neq -s^3$ As $r > 0$ (1)

$\therefore$ $4r^2 = s^2$

$2r = s$ As required.

Best Method: Subbing in

NB positive only as $r, s > 0$

let other roots be α & β

iii) If $s = 2r$ then $P(z)$ has a double root at $z = \frac{1}{2}$

$$\sum \alpha = \frac{1}{2} + \frac{1}{2} + \alpha + \beta$$

$$= \frac{s}{3r} \quad \left(\frac{-b}{a}\right)$$

$$= \frac{2}{3}$$

$$\therefore \quad \alpha + \beta = -\frac{1}{3}$$

$$\sum \alpha\beta\gamma\varepsilon = \frac{\alpha\beta}{4}$$

$$= r - s \quad \left(\frac{e}{a}\right)$$

$$= -r$$

$$\therefore \quad \alpha\beta = -4r$$

(1)

If $\alpha^2 + \beta^2 = 0$

then $(\alpha+\beta)^2 - 2\alpha\beta = 0$

$$\left(-\frac{1}{3}\right)^2 - 2(-4r) = 0$$

$$\frac{1}{9} + 8r = 0$$

$$r = -\frac{1}{72} \quad (1)$$

(c) (i) $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$ (1)

let $\cos\theta = x$

$$64x^6 - 112x^4 + 56x^2 - 7 = 0$$

becomes

$$64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7 = 0$$

from (1)

$$\frac{\cos 7\theta}{\cos\theta} = 0$$

$$\therefore \quad 7\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}$$

$$\theta = \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{\pi}{2}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}$$

$\therefore$ roots are $\cos\frac{\pi}{14}$, $\cos\frac{3\pi}{14}$, $\cos\frac{5\pi}{14}$, $\cos\frac{9\pi}{14}$, $\cos\frac{11\pi}{14}$, $\cos\frac{13\pi}{14}$

NB $\theta \neq \frac{\pi}{2}$ As $\cos\theta \neq 0$

$$\ln P(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx$$

ii)

Product of roots $= \dfrac{g}{a}$

$$\therefore \cos\frac{\pi}{14}\cdot\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}\cos\frac{9\pi}{14}\cos\frac{11\pi}{14}\cdot\cos\frac{13\pi}{14} = -\frac{7}{64}$$

NB//
$$\cos\frac{9\pi}{14} = \cos\left(\pi - \frac{5\pi}{14}\right) = -\cos\frac{5\pi}{14}$$
$$\cos\frac{11\pi}{14} = \cos\left(\pi - \frac{3\pi}{14}\right) = -\cos\frac{3\pi}{14}$$
$$\cos\frac{13\pi}{14} = \cos\left(\pi - \frac{\pi}{14}\right) = -\cos\left(\frac{\pi}{14}\right)$$

As they are in the second quadrant where 'cos' is negative.

$$\cos\frac{\pi}{14}\cdot\cos\frac{3\pi}{14}\cdot\cos\frac{5\pi}{14} \times -\cos\frac{\pi}{14}\cdot -\cos\frac{3\pi}{14}\cdot -\cos\frac{5\pi}{14} = -\frac{7}{64}$$

$$= \cancel{-}\left[\cos^2\frac{\pi}{14}\cdot\cos^2\frac{3\pi}{14}\cdot\cos^2\frac{5\pi}{14}\right] = \cancel{+}\frac{7}{64}$$

$$\therefore \cos\frac{\pi}{14}\cdot\cos\frac{3\pi}{14}\cdot\cos\frac{5\pi}{14} = \sqrt{\frac{7}{64}}$$

NB Positive only as all angles are in the first quadrant

(d) $$\int \frac{4x}{\sqrt{2x-3x^2}}\,dx$$

$$= 4\int \frac{x}{\sqrt{2x-3x^2}}\,dx$$

$$= \frac{4}{6}\int \frac{6x}{\sqrt{2x-3x^2}}\,dx$$

$$= \frac{2}{3}\int \frac{6x-2}{\sqrt{2x-3x^2}} + \frac{2}{\sqrt{2x-3x^2}}\,dx$$

(1) Approaching this received mark.

$$= -\frac{2}{3}\int \frac{2-6x}{\sqrt{2x-3x^2}}\,dx + \frac{4}{3}\int \frac{1}{\sqrt{2x-3x^2}}$$

$u = 2x - 3x^2$

$\dfrac{du}{dx} = 2 - 6x$

$$I = \int \frac{(2-6x)\,dx}{\sqrt{2x-3x^2}} = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + c$$

$$2x - 3x^2 = 3\left(\frac{2}{3}x - x^2\right)$$
$$= -3\left(x^2 - \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right)$$
$$= -3\left[\left(x-\frac{1}{3}\right)^2 - \frac{1}{9}\right]$$
$$= 3\left[\frac{1}{9} - \left(x-\frac{1}{3}\right)^2\right]$$

(1) Completing Square.

$$= -\frac{4}{3}\sqrt{2x-3x^2} + \frac{4}{3}\int \frac{1}{\sqrt{3\left[\frac{1}{9} - \left(x-\frac{1}{3}\right)^2\right]}}\,dx$$

$$= -\frac{4}{3}\sqrt{2x-3x^2} + \frac{4\sqrt{3}}{9}\int \frac{1}{\sqrt{\frac{1}{9}-\left(x-\frac{1}{3}\right)^2}}\,dx$$

$$= -\frac{4}{3}\sqrt{2x-3x^2} + \frac{4\sqrt{3}}{9}\sin\left[\frac{x-\frac{1}{3}}{\frac{1}{3}}\right] + c$$

$$= -\frac{4}{3}\sqrt{2x-3x^2} + \frac{4\sqrt{3}}{9}\sin\left[\frac{3x-1}{1}\right] + c$$

① Correct Solution

Question 15

(a)(i) If $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

then $\omega^7 = 1$ By DMT

$\therefore \omega^7 - 1 = 0$ ①

$(\omega-1)(\omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1) = 0$

As $\omega \neq 1$ ① By factorising

then $\omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1 = 0$

If they said that these are the roots of $\omega^7 - 1 = 0$ then they must find them first!

(ii) $a = -\sum \alpha$

$a = -(\omega+\omega^6+\omega^2+\omega^5+\omega^3+\omega^4)$

$= +1$ from (i)

$c = -\sum \alpha\beta\gamma$

$= -(\omega+\omega^6)(\omega^2+\omega^5)(\omega^3+\omega^4)$

$= (\omega^3 + \omega^6 + \omega^8 + \omega^{11})(\omega^3 + \omega^4)$

$= \omega^6 + \omega^9 + \omega^{11} + \omega^{14} + \omega^7 + \omega^{10} + \omega^{12} + \omega$

As $\omega^7 = 1$

$= \omega^6 + \omega^2 + \omega^4 + 1 + 1 + \omega^3 + \omega^5 + \omega$

$= 2 - 1$

$= 1$

$\therefore c = -1$

$b = \sum \alpha\beta$

$= (\omega + \omega^6)(\omega^2 + \omega^5) + (\omega^2 + \omega^5)(\omega^3 + \omega^4)$
$+ (\omega^3 + \omega^4)(\omega + \omega^6)$

$= \omega^3 + \omega^6 + \omega^8 + \omega^{11} + \omega^5 + \omega^6 + \omega^8 + \omega^9$
$+ \omega^4 + \omega^9 + \omega^5 + \omega^{10}$

As $\omega^7 = 1$

$= \omega^3 + \omega^6 + \cancel{\omega} + \cancel{\omega^4} + \cancel{\omega^5} + \omega^6 + \cancel{\omega} + \cancel{\omega^2}$
$\cancel{+\omega^4} + \cancel{\omega^2} + \cancel{\omega^5} + \cancel{\omega^3}$

$= 2(\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6)$

$= -2$

$x^3 + x^2 - 2x - 1 = 0$ has roots

$\omega + \omega^6, \quad \omega^2 + \omega^5, \quad \omega^3 + \omega^4$

iii) $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\omega^6 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}$ By DMT

$= \cos\frac{-2\pi}{7} + i\sin\frac{-2\pi}{7}$

As cosine is even & sine is odd

$= \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$

$= \overline{\omega}$ (1)

Using this method $\omega^5 = \overline{\omega^2}$

& $\omega^4 = \overline{\omega^3}$

$\therefore \omega + \omega^6 = 2\cos\frac{2\pi}{7}$

$\omega^2 + \omega^5 = 2\cos\frac{4\pi}{7}$

$\omega^3 + \omega^4 = 2\cos\frac{6\pi}{7}$ (1)

$$\therefore \sum \alpha = 2\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right] = -1$$

from (i)

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

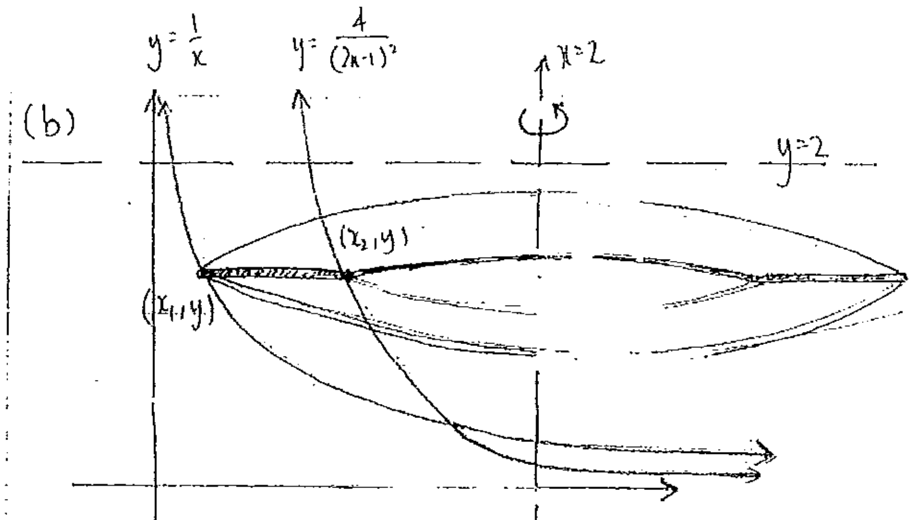
$$\cos\frac{4\pi}{7} = \cos\left(\pi - \frac{3\pi}{7}\right) = -\cos\frac{3\pi}{7}$$

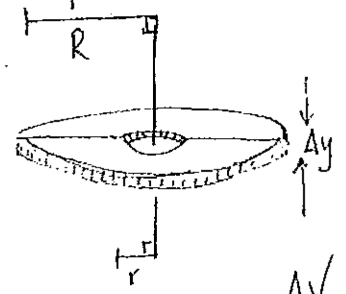
$$\cos\frac{6\pi}{7} = \cos\left(\pi - \frac{\pi}{7}\right) = -\cos\frac{\pi}{7} \quad (1)$$

As they are in the second quadrant.

$$\therefore \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$$

(b)





$$\Delta V = \pi\left(R^2 - r^2\right)\Delta y$$

$$\Delta V = \pi\left[(2-x_1)^2 - (2-x_2)^2\right]\Delta y \quad (A)$$

$$\Delta V = \pi\Big[\big[(2-x_1) - (2-x_2)\big]\big[(2-x_1) + (2-x_2)\big]\Big]\Delta y$$

$$x_1 = \frac{1}{y} \qquad y = \frac{4}{(2x_2-1)^2}$$

$$(2x_2-1)^2 = \frac{4}{y}$$

$$2x_2 - 1 = \frac{2}{\sqrt{y}}$$

$$2x_2 = \frac{2}{\sqrt{y}} + 1$$

$$x_2 = \frac{2+\sqrt{y}}{2\sqrt{y}}$$

$$\Delta V = \pi\left[\frac{2+\sqrt{y}}{2\sqrt{y}} - \frac{1}{y}\right]\left[4 - \left(\frac{2+\sqrt{y}}{2\sqrt{y}} + \frac{1}{y}\right)\right]\Delta y$$ B

C

$$V \doteq \pi \sum_{y=2(2-\sqrt{3})}^{2} \left[\frac{2+\sqrt{y}}{2\sqrt{y}} - \frac{1}{y}\right]\left[4 - \left(\frac{2+\sqrt{y}}{2\sqrt{y}} + 1/y\right)\right]\Delta y$$

$$= \lim_{\Delta y \to 0} \pi \sum_{y=2(2-\sqrt{3})}^{2} \left[\frac{2+\sqrt{y}}{2\sqrt{y}} - \frac{1}{y}\right]\left[4 - \left(\frac{2+\sqrt{y}}{2\sqrt{y}} + \frac{1}{y}\right)\right]\Delta y$$

$$= \pi\int_{2(2-\sqrt{3})}^{2} \left(\frac{2+\sqrt{y}}{2\sqrt{y}} - \frac{1}{y}\right)\left[4 - \left(\frac{2+\sqrt{y}}{2\sqrt{y}} + \frac{1}{y}\right)\right] dy$$

$$= \int_{2(2-\sqrt{3})}^{2} \pi\left(\frac{2\sqrt{y}+y-2}{2y}\right)\left(4 - \frac{2\sqrt{y}+y+2}{2y}\right) dy$$

$$= \pi\int_{2(2-\sqrt{3})}^{2} \frac{(2\sqrt{y}+y-2)(8y-2\sqrt{y}-y-2)}{4y^2}\, dy$$

$$= \pi\int_{2(2-\sqrt{3})}^{2} \frac{(2\sqrt{y}+y-2)(7y-2\sqrt{y}-2)}{4y^2}\, dy$$

$$= \pi\int_{2(2-\sqrt{3})}^{2} \frac{14y\sqrt{y} - 4y - 4\sqrt{y} + 7y^2 - 2y\sqrt{y} - 2y - 14y + 4\sqrt{y} \cdots}{4y^2}$$

$$= \pi\int_{2(2-\sqrt{3})}^{2} \frac{12y\sqrt{y} - 20y + 7y^2 + 4}{4y^2}\, dy$$

$$= \pi\int_{2(2-\sqrt{3})}^{2} 3y^{-\frac{1}{2}} - 5y^{-1} + \frac{7}{4} + y^{-2}\, dy$$

$$= \pi\left[6y^{\frac{1}{2}} - 5\ln y + \frac{7y}{4} - \frac{1}{y}\right]_{2(2-\sqrt{3})}^{2}$$

$$= \pi\left[(8.01954) - (6.58315)\right]$$ D

$= 4.52$ units^3 (2dp)

1 Correct dimensions of typical slice A

1 Correct expression in terms of y B

1 Correct sum C

1 Correct answer + working D

(c) Equation of tangent @ P

$$x + p^2 y - 2cp = 0 \quad ①$$

@ Q $\quad x + q^2 y - 2cq = 0 \quad ②$

T is the point of intersection

① - ② $\quad (p^2 - q^2) y - 2cp + 2cq = 0$

$$(p+q)\cancel{(p-q)}\, y = 2c\cancel{(p-q)}$$

$$y = \frac{2c}{p+q} \quad ③$$

③ ⇒ ①

$$x + p^2\left(\frac{2c}{p+q}\right) - 2cp = 0$$

$$x = 2cp - \frac{2cp^2}{p+q}$$

$$= \frac{2cp(p+q) - 2cp^2}{p+q}$$

$$= \frac{\cancel{2cp^2} + 2cpq - \cancel{2cp^2}}{p+q}$$

$$= \frac{2cpq}{p+q}$$

$$T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right) \quad ①$$

If PQ passes through pq we need the equation of chord PQ

$$m_{PQ} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$$

$$= \frac{cq - cp}{pq} \div c(p-q)$$

$$= \frac{-\cancel{c}\cancel{(p-q)}}{pq} \times \frac{1}{\cancel{c}\cancel{(p-q)}}$$

$$= -\frac{1}{pq}$$

$$-\frac{1}{pq} = \frac{y - \frac{c}{p}}{x - cp}$$

$$-\frac{x}{pq} + \frac{c}{q} = y - \frac{c}{p}$$

$$-\frac{x}{pq} - y + c\left(\frac{1}{p} + \frac{1}{q}\right) = 0$$

$$\frac{x}{pq} + y = c\left(\frac{p+q}{pq}\right)$$

When $y=0$, $x = pq$

$$\frac{\cancel{pq}}{\cancel{pq}} = c\left(\frac{p+q}{pq}\right)$$

(1)

$$pq = c(p+q) \quad (4)$$

Locus of T

$$x = \frac{2cpq}{p+q} \qquad y = \frac{2c}{p+q}$$

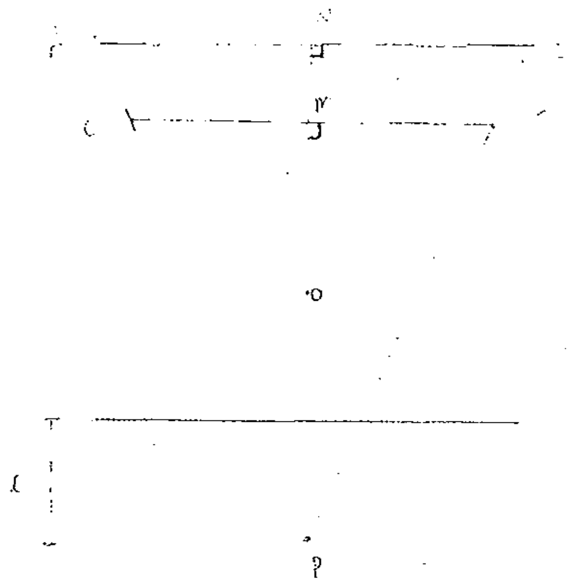
from (4)

$$x = \frac{2c^2\cancel{(p+q)}}{\cancel{p+q}}$$

$$= 2c^2$$ (1)

$\therefore T$ lies on the line $x = 2c^2$

Question 16



(i) Using ΔPQB & ΔPMD

$\Delta PQB \;|||\; \Delta PMD$ (equiangular)

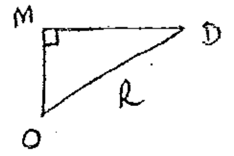
$$\therefore \frac{QB}{PQ} = \frac{MD}{MP}$$

From information given $QB = R$ $MP = OM + OP$

$PQ = 2R$ $OP = R$

$$\therefore \frac{R}{2R} = \frac{MD}{OM+R}$$ (1) Similar Δ's

from $\triangle ODM$



$$DM^2 = R^2 - OM^2$$

$$\therefore \left(\frac{1}{2}\right)^2 = \frac{R^2 - OM^2}{(R + OM)^2}$$

$$R^2 + 2R.OM + OM^2 = 4R^2 - 4OM^2 \quad (1)$$

$$5.OM^2 + 2R.OM - 3R^2 = 0$$

This is a quadratic in OM

$$(5.OM - 3R)(OM + R) = 0$$

Producing and solving an equation with only OM & R

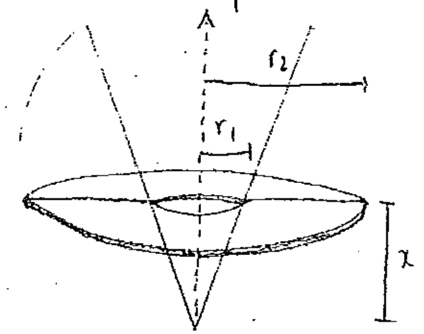
As $OM > 0$ $\quad 5.OM - 3R = 0$

$$5.OM = 3R$$

$$OM = \frac{3}{5}R$$

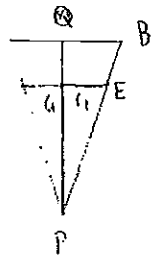
As required.

(b) Each piece is a disc



$$\Delta V = \pi\left[r_2^2 - r_1^2\right]\Delta x$$

(1)



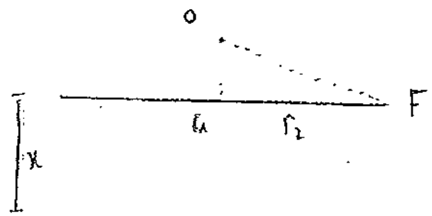
let the centre the piece be G and the inside edge be E

As $\triangle PQB \mathbin{|||} \triangle PGE$

$$\frac{PQ}{QB} = \frac{PG}{GE}$$

$$\frac{2R}{R} = \frac{x}{r_1}$$

$$\therefore r_1 = \frac{x}{2}$$



let the outside edge of the slice be F

In $\triangle OGF$

$$OF^2 = OG^2 + GF^2$$

$$\therefore R^2 = (R - x)^2 + (r_2)^2$$

$$(r_2)^2 = R^2 - (R - x)^2$$

$$= \left[R - (R - x)\right]\left[R + (R - x)\right]$$

$$= x(2R - x)$$

(1) Relating R_1 & r_2 to x

$$\therefore \Delta V = \pi\left(x(2R-x) - \frac{x^2}{4}\right)\Delta x$$

$$= \pi\left(2Rx - x^2 - \frac{x^2}{4}\right)\Delta x \quad (1)$$

$$= \pi\left(2Rx - \frac{5x^2}{4}\right)\Delta x \quad \text{as}$$

required.

(b) $$I_n = \int_0^{2\pi} (1+\cos x)^n \, dx$$

$$I_{n+1} = \int_0^{2\pi} (1+\cos x)^{n+1} \, dx$$

$$= \int_0^{2\pi} (1+\cos x)(1+\cos x)^{n} \, dx$$

$$= \int_0^{2\pi} (1+\cos x)^n \, dx + \int_0^{2\pi} \cos x\,(1+\cos$$

$$= I_n + \int_0^{2\pi} \cos x\,(1+\cos x)^n \, dx \quad (1)$$

let $$J_n = \int_0^{2\pi} \cos x\,(1+\cos x)^n \, dx$$

$$u = (1+\cos x)^n \qquad v = \sin x$$

$$\frac{du}{dx} = -n(1+\cos x)^{n-1}\sin x \qquad \frac{dv}{dx} = \cos x \quad (1)$$

Any use of 'by parts' that is going toward the answer.

$$\therefore J_n = \left[(\cos x + 1)^n \sin x \right]_0^{2\pi}$$

$$+ n \int_0^{2\pi} \sin^2 x \, (1 + \cos x)^{n-1} \, dx$$

$$= [0 - 0] + n \int_0^{2\pi} (1 - \cos^2 x)(1 + \cos x)^{n-1} d$$

$$= n \int_0^{\pi} (1 + \cos x)(1 - \cos x)(1 + \cos x)^{n-1} d$$

$$= n \int_0^{2\pi} (1 - \cos x)(1 + \cos x)^n \, dx$$

$$= n \int_0^{2\pi} (1 + \cos x)^n \, dx$$

$$- n \int_0^{2\pi} \cos x \, (1 + \cos x)^n \, dx$$

$$J_n = n I_n - n J_n \quad (1)$$

$$(1+n) J_n = n I_n \quad , \quad J_n = \frac{n I_n}{(n+1)}$$

$$\therefore I_{n+1} = I_n + \frac{n I_n}{n+1}$$

$$I_{n+1} = \frac{(n+1) I_n + n I_n}{n+1}$$

$$= \frac{(n+1+n) I_n}{n+1}$$

(1) Finishing the proof.

$$= \frac{2n+1}{n+1} I_n$$

As required.

(c) $y = \ln x$

(i) $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}$$

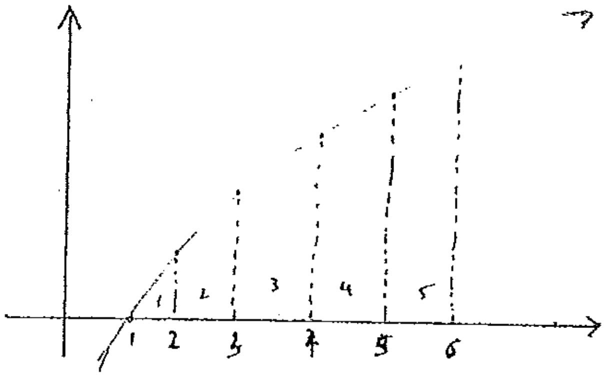
$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

$-\frac{1}{x^2}$ is always less than zero as $x^2 \geq 0$

$\therefore$ $\frac{d^2y}{dx^2}$ is always negative .. (1)

$y = \ln x$ is always concave down

ii)



$$A = \frac{h}{2}(a+b)$$

(1) $A = \frac{1}{2}(\ln(1) + \ln(2))$

(2) $A = \frac{1}{2}(\ln(2) + \ln(3))$

(3) $A = \frac{1}{2}(\ln(3) + \ln(4))$

...

(n-1) $A = \frac{1}{2}(\ln(n-1) + \ln(n))$

$\therefore$ Total Area $= \frac{1}{2}\Big[\ln(1) + \ln(n) + 2(\ln 2 + \ln 3 + \dots \ln(n-1))\Big]$

let $A_1 =$ (1)

iii) $A_2 = \int_1^n \ln x \, dx$ (1)

$u = \ln x \qquad v = x$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = 1$

$$= \Big[x \ln x\Big]_1^n - \int_1^n 1 \, dx$$

$$= \Big[n \ln n - 0\Big] - \Big[x\Big]_1^n$$

$$= n \ln n - n + 1$$

$$= 1 + n(\ln(n) - 1)$$ (1)

iv) As the function is concave down, the area found in ii) is slightly less than the actual area. 1

1

$A_1 < A_2$

$A_1 = \frac{1}{2}\left[\ln(1) + \ln(n) + 2\left(\ln(2) + \ln(3) + \dots + \ln(n-1\right.\right.$

$= \frac{1}{2}\left\{2\ln(1) + 2\ln(n) + \left(\quad\right) - \ln(1) - \ln(n\right.$

$= \frac{1}{2}\left[2\left[\ln(1) + \ln(2) + \dots + \ln(n-1) + \ln(n)\right] - \ln(1) - \ln(n)\right]$

$= \frac{1}{2}\left[2\left[\ln\left[1 \times 2 \times 3 \times \dots \times (n-1)(n)\right]\right] - \ln(1) - \ln(n)\right]$

$= \frac{1}{2}\left[2\ln(n!) - 0 - \ln(n)\right]$

$= \ln(n!) - \frac{1}{2}\ln(n)$

$\therefore \ln(n!) - \frac{1}{2}\ln(n) < 1 - n + n(\ln(n))$

$\ln(n!) < 1 - n + \left(n + \frac{1}{2}\right)\ln(n)$

$\ln(n!) < 1 - n + \ln\left[n^{(n+\frac{1}{2})}\right]$

$\therefore e^{\ln(n!)} < e^{\left[(1-n) + \ln\left[n^{(n+\frac{1}{2})}\right]\right]}$

$n! < \frac{e^1 \times e^{\ln\left[n^{(n+\frac{1}{2})}\right]}}{e^n}$

$< \frac{e \cdot n^{(n+\frac{1}{2})}}{e^n}$ As required.

1 Showing all working.