NSW EDUCATION STANDARDS AUTHORITY

2017 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which polynomial is a factor of $x^3 - 5x^2 + 11x - 10$?

A. $x - 2$

B. $x + 2$

C. $11x - 10$

D. $x^2 - 5x + 11$

2 It is given that $\log_a 8 = 1.893$, correct to 3 decimal places.

What is the value of $\log_a 4$, correct to 2 decimal places?

A. 0.95

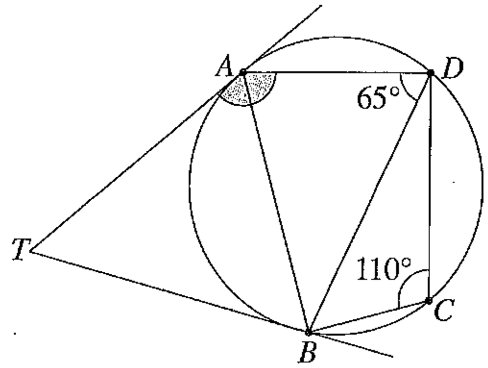
B. 1.26

C. 1.53

D. 2.84

3 The points A, B, C and D lie on a circle and the tangents at A and B meet at T, as shown in the diagram.

The angles BDA and BCD are 65° and 110° respectively.



What is the value of $\angle TAD$?

A. 130°

B. 135°

C. 155°

D. 175°

4 What is the value of $\tan\alpha$ when the expression $2\sin x - \cos x$ is written in the form $\sqrt{5}\sin(x-\alpha)$?

A. -2

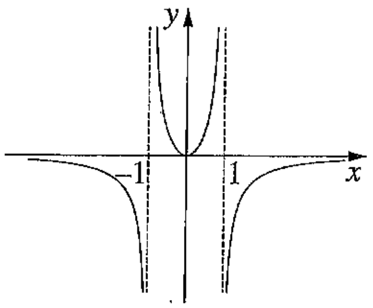
B. $-\dfrac{1}{2}$

C. $\dfrac{1}{2}$

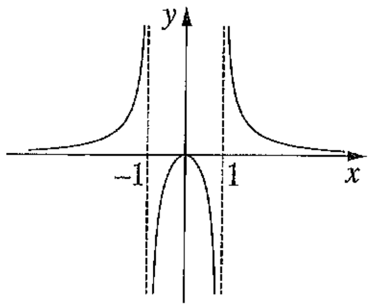
D. 2

5 Which graph best represents the function $y = \dfrac{2x^2}{1 - x^2}$?

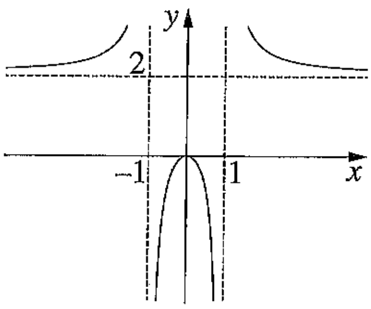
A.



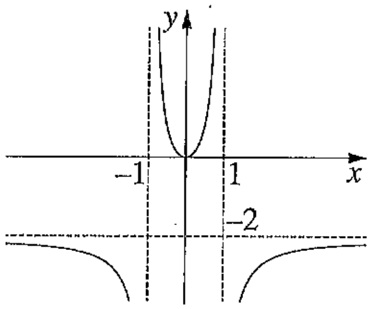
B.



C.



D.



6 The point $P\left(\dfrac{2}{p}, \dfrac{1}{p^2}\right)$, where $p \neq 0$, lies on the parabola $x^2 = 4y$.

What is the equation of the normal at P?

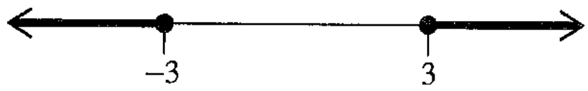
A. $py - x = -p$

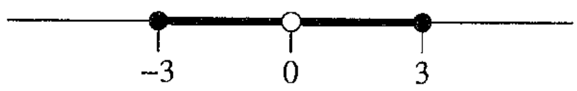
B. $p^2y + px = -1$

C. $p^2y - p^3x = 1 - 2p^2$

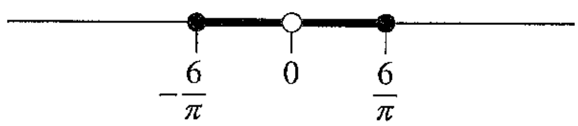
D. $p^2y + p^3x = 1 + 2p^2$

7 Which diagram represents the domain of the function $f(x) = \sin^{-1}\left(\frac{3}{x}\right)$?

A. 

B. 

C. 

D. 

8 A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of 5 cm s^{-1}.

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. 25π cm^2 s^{-1}

B. 30π cm^2 s^{-1}

C. 150π cm^2 s^{-1}

D. 225π cm^2 s^{-1}

9 When expanded, which expression has a non-zero constant term?

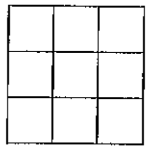
A. $\left(x + \frac{1}{x^2}\right)^7$

B. $\left(x^2 + \frac{1}{x^3}\right)^7$

C. $\left(x^3 + \frac{1}{x^4}\right)^7$

D. $\left(x^4 + \frac{1}{x^5}\right)^7$

10 Three squares are chosen at random from the 3×3 grid below, and a cross is placed in each chosen square.



What is the probability that all three crosses lie in the same row, column or diagonal?

A. $\frac{1}{28}$

B. $\frac{2}{21}$

C. $\frac{1}{3}$

D. $\frac{8}{9}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval from $A(-4,-4)$ to $B(1,6)$ internally in the ratio $2:3$. **1**

Find the x-coordinate of P.

(b) Differentiate $\tan^{-1}(x^3)$. **2**

(c) Solve $\dfrac{2x}{x+1} > 1$. **3**

(d) Sketch the graph of the function $y = 2\cos^{-1}x$. **2**

(e) Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$. **3**

(f) Find $\displaystyle\int \sin^2 x \cos x\,dx$. **1**

Question 11 continues on page 8

(g) The probability that a particular type of seedling produces red flowers is $\frac{1}{5}$. Eight of these seedlings are planted.

(i) Write an expression for the probability that exactly three of the eight seedlings produce red flowers. **1**

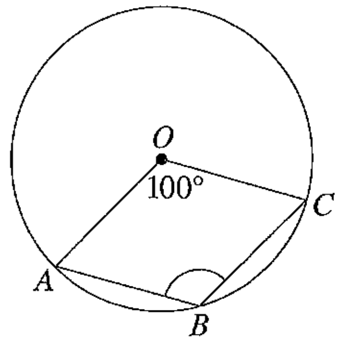
(ii) Write an expression for the probability that none of the eight seedlings produces red flowers. **1**

(iii) Write an expression for the probability that at least one of the eight seedlings produces red flowers. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle AOC$ is $100°$. **2**



NOT TO SCALE

Find the size of $\angle ABC$, giving reasons.

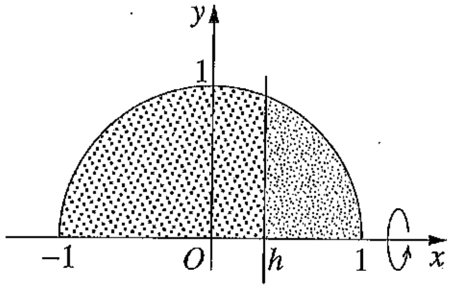
(b) (i) Carefully sketch the graphs of $y = |x+1|$ and $y = 3 - |x-2|$ on the same axes, showing all intercepts. **3**

(ii) Using the graphs from part (i), or otherwise, find the range of values of x for which **1**

$$|x+1| + |x-2| = 3.$$

Question 12 continues on page 10

(c) The region enclosed by the semicircle $y = \sqrt{1 - x^2}$ and the x-axis is to be divided into two pieces by the line $x = h$, where $0 \le h < 1$.



The two pieces are rotated about the x-axis to form solids of revolution. The value of h is chosen so that the volumes of the solids are in the ratio 2:1.

(i) Show that h satisfies the equation $3h^3 - 9h + 2 = 0$. **3**

(ii) Given $h_1 = 0$ as the first approximation for h, use one application of Newton's method to find a second approximation for h. **1**

(d) At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$. **3**

Find the acceleration of the particle as a function of x.

(e) Evaluate $\lim_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the x-axis in simple harmonic motion centred at the origin. **3**

When $x = 2$ the velocity of the particle is 4.

When $x = 5$ the velocity of the particle is 3.

Find the period of the motion.

(b) Let n be a positive EVEN integer.

(i) Show that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$. **2**

(ii) Hence show that **1**

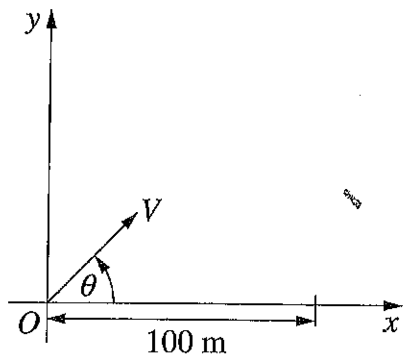
$$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right].$$

(iii) Hence show that $\binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n} = n2^{n-3}$. **2**

Question 13 continues on page 12

Question 13 (continued)

(c) A golfer hits a golf ball with initial speed V m s^{-1} at an angle θ to the horizontal. The golf ball is hit from one side of a lake and must have a horizontal range of 100 m or more to avoid landing in the lake.



Neglecting the effects of air resistance, the equations describing the motion of the ball are

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time in seconds after the ball is hit and g is the acceleration due to gravity in m s^{-2}. Do NOT prove these equations.

(i) Show that the horizontal range of the golf ball is $\dfrac{V^2 \sin 2\theta}{g}$ metres. **2**

(ii) Show that if $V^2 < 100g$ then the horizontal range of the ball is less than 100 m. **1**

It is now given that $V^2 = 200g$ and that the horizontal range of the ball is 100 m or more.

(iii) Show that $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$. **2**

(iv) Find the greatest height the ball can achieve. **2**

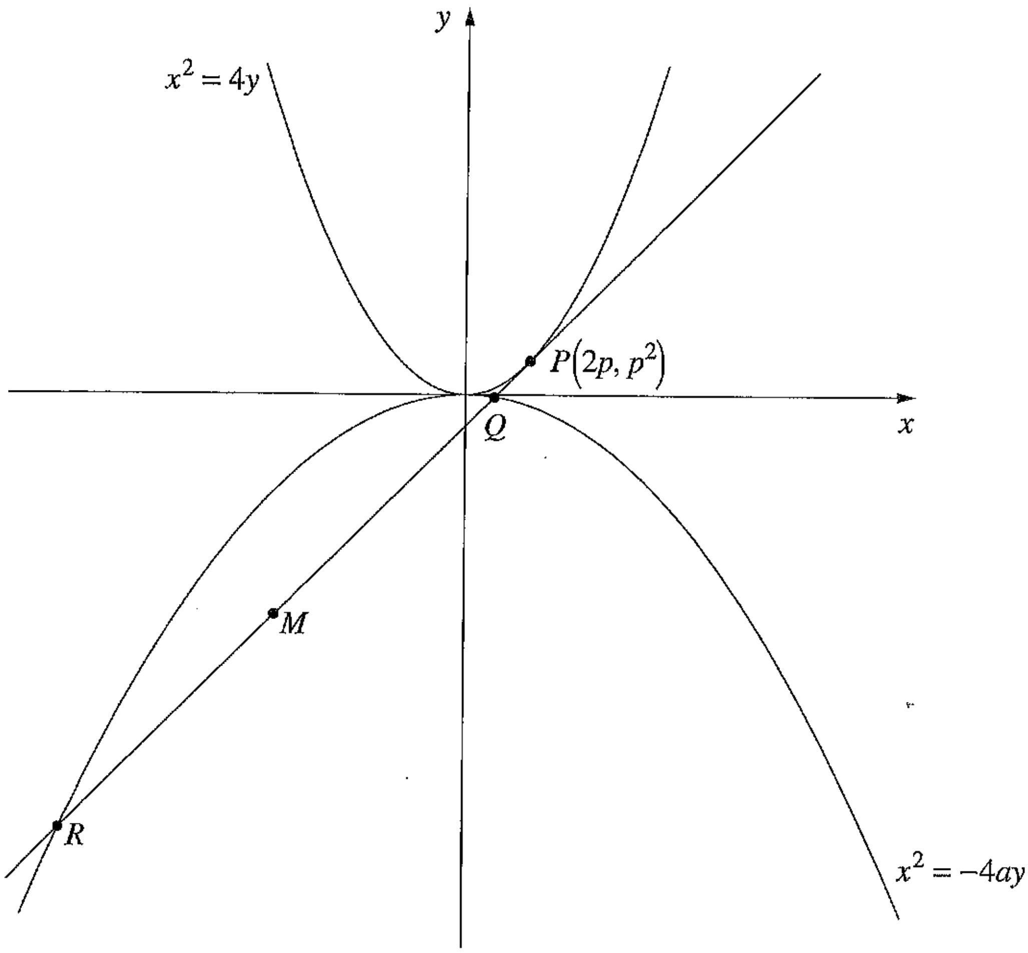
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that $8^{2n+1} + 6^{2n-1}$ is divisible by 7, for any integer $n \geq 1$. **3**

(b) Let $P(2p, p^2)$ be a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P meets the parabola $x^2 = -4ay$, $a > 0$, at Q and R. Let M be the midpoint of QR.



(i) Show that the x coordinates of R and Q satisfy **2**

$$x^2 + 4apx - 4ap^2 = 0.$$

(ii) Show that the coordinates of M are $\left(-2ap, -p^2(2a+1)\right)$. **2**

(iii) Find the value of a so that the point M always lies on the parabola $x^2 = -4y$. **2**

Question 14 continues on page 14

(c) The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

(i) By differentiating the product $F(t)e^{0.4t}$ show that **2**

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. **2**

(iii) The concentration of the drug increases to a maximum. **2**

For what value of t does this maximum occur?

End of paper

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2017 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	B
3	B
4	C
5	D
6	D
7	A
8	C
9	C
10	B

Section II

Question 11 (a)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$x \text{ value of } P = \frac{2\times 1+3\times -4}{2+3}$$

$$= -\frac{10}{5}$$

$$= -2$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the chain rule, or equivalent merit	1

Sample answer:

Let $y = \tan^{-1}\left(x^3\right)$

$$y' = \frac{1}{1+\left(x^3\right)^2}\times 3x^2$$

$$= \frac{3x^2}{1+x^6}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Correctly identifies 1 and –1 as important, or equivalent merit	2
• Attempts to deal with the denominator, or equivalent merit	1

Sample answer:

$$\frac{2x}{x+1} > 1$$

Multiply both sides by $(x + 1)^2$

$$2x(x+1) > (x+1)^2$$

$$2x^2 + 2x > x^2 + 2x + 1$$

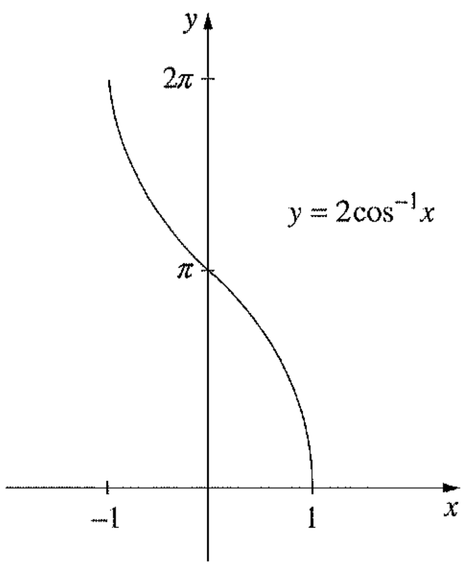
$$x^2 > 1$$

$$\therefore\ x > 1 \quad \text{or} \quad x < -1$$

Question 11 (d)

Criteria	Marks
• Provides correct sketch	2
• Indicates correct range, or equivalent merit	1

Sample answer:



Question 11 (e)

Criteria	Marks
• Provides correct solution	3
• Provides a correct primitive, or equivalent merit	2
• Attempts to use given substitution, or equivalent merit	1

Sample answer:

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx \qquad x = u^2 - 1$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u\,du$$

when $x = 0$ $\quad u^2 - 1 = 0 \quad u = 1$

when $x = 3$ $\quad u^2 - 1 = 3 \quad u = 2$

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u^2 - 1}{u}.2u\,du$$

$$= 2\int_1^2 (u^2 - 1)\,du$$

$$= 2\left[\frac{u^3}{3} - u\right]_1^2$$

$$= 2\left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right]$$

$$= \frac{8}{3}$$

Question 11 (f)

Criteria	Marks
• Provides correct primitive	1

Sample answer:

$$\int \sin^2 x \cos x\,dx$$

$$= \frac{\sin^3 x}{3} + c$$

Question 11 (g) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\binom{8}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^5$$

Question 11 (g) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\left(\frac{4}{5}\right)^8$$

Question 11 (g) (iii)

Criteria	Marks
• Provides correct answer	1

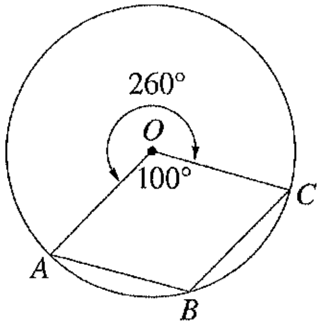
Sample answer:

$$1-\left(\frac{4}{5}\right)^8$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	2
• Attempts to relate two angles in the diagram, or equivalent merit	1

Sample answer:



Reflex $\angle AOC = 360° - 100°$ (angle of revolution)

$= 260°$

$\angle ABC = \frac{1}{2}$ reflex $\angle AOC$ (angle at centre equals twice the angle at circumference standing in the same arc)

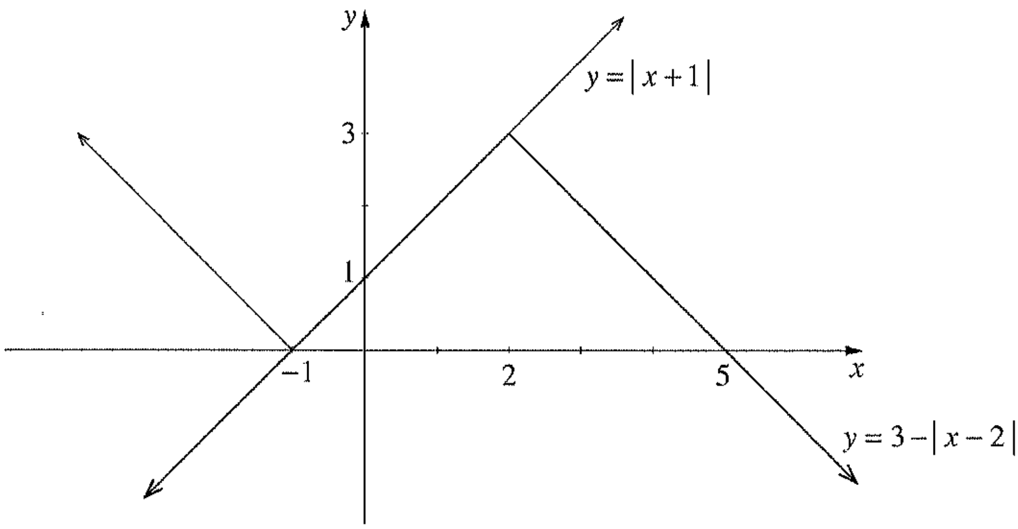
$= \frac{1}{2}(260°)$

$= 130°$

Question 12 (b) (i)

Criteria	Marks
• Provides correct sketch	3
• Provides correct sketch of $y = 3 - \|x - 2\|$, or equivalent merit	2
• Provides correct sketch of $y = \|x + 1\|$, or equivalent merit	1

Sample answer:



Question 12 (b) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$\therefore\ -1 \le x \le 2$

when $|x+1| = 3 - |x-2|$ the graphs have points in common.

ie $|x+1| + |x-2| = 3$

Question 12 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Equates the volumes in the correct ratio and attempts to evaluate an integral, or equivalent merit	2
• Provides correct integral for the volume of one solid, or equivalent merit	1

Sample answer:

$$V_1 = \pi \int_h^1 y^2 \, dx$$

$$= \pi \int_h^1 1 - x^2 \, dx$$

$$= \pi \left[x - \frac{x^3}{3} \right]_h^1$$

$$= \pi \left[1 - \frac{1}{3} \right] - \pi \left[h - \frac{h^3}{3} \right]$$

$$= \pi \left[\frac{2}{3} - h + \frac{h^3}{3} \right]$$

$$V_2 = \pi \int_{-1}^h 1 - x^2 \, dx$$

$$= \pi \left[x - \frac{x^3}{3} \right]_{-1}^h$$

$$= \pi \left[h - \frac{h^3}{3} \right] - \pi \left[-1 - \frac{-1}{3} \right]$$

$$= \pi \left[h - \frac{h^3}{3} + \frac{2}{3} \right]$$

Ratio $V_2 : V_1 = 2 : 1 \qquad \therefore V_2 = 2V_1$

$$h - \frac{h^3}{3} + \frac{2}{3} = 2\left(\frac{2}{3} - h + \frac{h^3}{3} \right)$$

$$= \frac{4}{3} - 2h + \frac{2h^3}{3}$$

$\therefore \quad 3h - h^3 + 2 = 4 - 6h + 2h^3$

$\therefore \; 3h^3 - 9h + 2 = 0$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

Let $f(h) = 3h^3 - 9h + 2 \qquad h_1 = 0$

$$h_2 = h_1 - \frac{f(h_1)}{f'(h_1)} = 0 - \frac{f(0)}{f'(0)}$$

$$f(h_1) = f(0) = 2$$

$$f'(h) = 9h^2 - 9$$

$$\therefore f'(0) = 9(0)^2 - 9$$
$$= -9$$

$$\therefore \quad h_2 = 0 - \frac{2}{-9}$$
$$= \frac{2}{9}$$

Question 12 (d)

Criteria	Marks
• Provides correct solution	3
• Attempts to use $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$, or equivalent merit	2
• Correctly finds $\dfrac{dt}{dx}$, or equivalent merit	1

Sample answer:

$$t = 4 - e^{-2x}$$

$$\frac{dt}{dx} = +2e^{-2x}$$

$$\therefore \frac{dx}{dt} = \frac{1}{2}e^{2x}$$

$$v = \frac{1}{2}e^{2x}$$

$$v^2 = \frac{1}{4}e^{4x}$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{8}e^{4x}\right)$$

$$= \frac{4}{8}e^{4x}$$

$$\ddot{x} = \frac{1}{2}e^{4x}$$

Question 12 (e)

Criteria	Marks
• Provides correct solution	2
• Correctly uses double angle result, or equivalent merit	1

Sample answer:

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \text{ since } \cos 2x = 1 - 2\sin^2 x$$

$$= 2\lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

$$= 2\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2$$

$$= 2 \times 1^2$$

$$= 2$$

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Correctly finds the value of n^2, or equivalent merit	2
• Correctly uses $v^2 = n^2\left(a^2 - x^2\right)$, or equivalent merit	1

Sample answer:

$$v^2 = n^2\left(a^2 - x^2\right)$$

$\therefore \quad 4^2 = n^2a^2 - 2^2n^2$ ——①

and $\quad 3^2 = n^2a^2 - 5^2n^2$ ——②

①–② $\therefore 7 = -4n^2 + 25n^2$

$$21n^2 = 7$$

$$n^2 = \frac{1}{3}$$

$$n = \frac{1}{\sqrt{3}}, \quad n > 0$$

Period, $T = \dfrac{2\pi}{n}$

$$= \frac{2\pi}{\frac{1}{\sqrt{3}}}$$

$$= 2\pi\sqrt{3}$$

Question 13 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly applies binomial theorem to one binomial expression, or equivalent merit	1

Sample answer:

n is even

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n$$

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + (-1)^r\binom{n}{r}x^2 + \cdots + \binom{n}{n}x^n$$

$$(1-x)^n + (1-x)^n = 2\binom{n}{0} + 2\binom{n}{2}x^2 + \cdots + 2\binom{n}{n}x^n$$

(odd terms have different sign and so cancel)

$$(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \cdots + \binom{n}{n}x^n\right]$$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Differentiating with respect to x.

$$n(1+x)^{n-1} - n(1-x)^{n-1} = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right]$$

$$n\left((1+x)^{n-1} - (1-x)^{n-1}\right) = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right]$$

Question 13 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Uses $x = 1$, or equivalent merit	1

Sample answer:

Let $x = 1$

$$n\left(2^{n-1} - 0\right) = 2\left[2\binom{n}{2} + 4\binom{n}{4} + \cdots + n\binom{n}{n}\right]$$

$$n\,2^{n-2} = 2\binom{n}{2} + 4\binom{n}{4} + \cdots + n\binom{n}{n}$$

$$n\,2^{n-3} = \binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n}$$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to solve $y = 0$, or equivalent merit	1

Sample answer:

Set $0 = y = Vt\sin\theta - \frac{1}{2}gt^2$

$t = 0$ or $V\sin\theta = \frac{1}{2}gt$

$$t = \frac{2V\sin\theta}{g}$$

At this value

$$x = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta$$

$$= \frac{V^2}{g}\sin 2\theta$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\text{Range} = \frac{V^2}{g}\sin 2\theta$$

If $v^2 < 100g$, then range $\frac{100g\sin 2\theta}{g}$

ie range $< 100\sin 2\theta$

since $\sin 2\theta \le 1$

then range < 100

Question 13 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Provides an inequality in $\sin 2\theta$, or equivalent merit	1

Sample answer:

If $V^2 = 200g$, then range $= \frac{200g\sin 2\theta}{g} \ge 100$

ie $200\sin 2\theta \ge 100$

$$\sin 2\theta \ge \frac{1}{2}$$

Now, $0 \le \theta \le \frac{\pi}{2}$ or $0 \le 2\theta \le \pi$

$$\frac{\pi}{6} \le 2\theta \le \frac{5\pi}{6}$$

$$\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$$

Question 13 (c) (iv)

Criteria	Marks
• Provides correct solution	2
• Obtains correct expression for the maximum height, or equivalent merit	1

Sample answer:

Maximum height occurs when

$$0 = \frac{dy}{dt} = V\sin\theta - gt$$

$$t = \frac{V}{g}\sin\theta$$

At this time

$$y = \frac{V^2}{g}\sin^2\theta - \frac{1}{2}g\frac{V^2}{g^2}\sin^2\theta$$

$$= \frac{V^2}{g}\frac{\sin^2\theta}{2}$$

$$= 100\sin^2\theta \qquad \left(\frac{V^2}{g} = 200\right)$$

For $0 \le \theta \le \frac{\pi}{2}$ $\sin^2\theta$ is an increasing function,

so maximum occurs when $\theta = \frac{5\pi}{12}$

$\therefore$ greatest height is $100\sin^2\left(\frac{5\pi}{12}\right)$

Question 14 (a)

Criteria	Marks
• Provides correct solution	3
• Correctly shows $P(n) \Rightarrow P(n+1)$, or equivalent merit	2
• Establishes result for $n = 1$, or equivalent merit	1

Sample answer:

Let $P(n)$ be the given proposition.

$P(1)$ is true since $8^3 + 6 = 518 = 7 \times 74$ which is divisible by 7.

Let k be an integer for which $P(k)$ is true.

That is $8^{2k+1} + 6^{2k-1} = 7m$, for some integer m.

Consider $8^{2(k+1)+1} + 6^{2(k+1)-1}$

$= 8^2 \times 8^{2k+1} + 6^2 \times 6^{2k-1}$

$= 8^2 \times 8^{2k+1} + 8^2 \times 6^{2k-1} - 8^2 \times 6^{2k-1} + 6^2 \times 6^{2k-1}$

$= 8^2\left(8^{2k+1} + 6^{2k-1}\right) + \left(6^2 - 8^2\right)6^{2k-1}$

$= 64 \times 7m - 28 \times 6^{2k-1}$

$= 7\left(64m - 4 \times 6^{2k-1}\right)$

which is divisible by 7, since m and k are integers and $64m - 4 \times 6^{2k-1}$ is an integer.

$\therefore$ $P(k + 1)$ is also true

$\therefore$ $P(n)$ is true for all $n \geq 1$ by induction.

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains equation of tangent at P, or equivalent merit	1

Sample answer:

The equation of the tangent at P is

$$y = px - p^2$$

substituting into $x^2 = -4ay$

$$-\frac{x^2}{4a} = px - p^2$$

$\therefore\ x^2 + 4apx - 4ap^2 = 0$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct coordinates	2
• Correctly verifies one coordinate	1

Sample answer:

The sum of the roots of the equation is $-4ap$.
x-coordinate of M is average of the roots
$\therefore$ x-coordinate of M is $-2ap$.

Substituting into the tangent,

$y = p \times -2ap - p^2$

$= -p^2(2a+1)$

$\therefore M$ is the point $\left(-2ap, -p^2(2a+1)\right)$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Eliminates the parameter, or equivalent merit	1

Sample answer:

Let $x = -2ap$, $y = -p^2(2a+1)$

$$\therefore y = -\left(\frac{x}{2a}\right)^2\left(2a+1\right)$$

$$= -x^2\frac{(2a+1)}{4a^2}$$

so $$x^2 = -\frac{4a^2}{2a+1}y$$

Hence $$\frac{a^2}{2a+1} = 1$$

ie $a^2 - 2a - 1 = 0$

$$\therefore a = 1 \pm \sqrt{2}$$

$$\therefore a = 1 + \sqrt{2},\ \text{since } a > 0$$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly applies the product rule, or equivalent merit	1

Sample answer:

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = F'(t)e^{0.4t} + 0.4F(t)e^{0.4t}$$

$$= e^{0.4t}\left(50e^{-0.5t} - 0.4F(t)\right) + 0.4F(t)e^{0.4t}$$

$$= 50e^{-0.1t}$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly integrates the expression given in part (i), or equivalent merit	1

Sample answer:

Integrating,

$$F(t)e^{0.4t} = \int 50e^{-0.1t}\,dt$$

$$= -500e^{-0.1t} + c$$

Now $F(t) = 0$ when $t = 0$ $\therefore$ $c = +500$

$$\therefore \quad F(t)e^{0.4t} = 500\left(1 - e^{-0.1t}\right)$$

$$F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$$

Question 14 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Makes substantial progress	1

Sample answer:

$$F'(t) = 500\left(-0.4e^{-0.4t} + 0.5e^{-0.5t}\right)$$

$$= 0 \text{ for a maximum}$$

$$\therefore 0.4e^{-0.4t} = 0.5e^{-0.5t}$$

$$\therefore \frac{5}{4} = e^{0.1t}$$

$$t = 10\ln\frac{5}{4}$$

$$(\simeq 2.23 \text{ hours})$$

Alternative

$$F'(t) = 50e^{-0.5t} - 0.4F(t)$$

$$= 50e^{-0.5t} - 0.4\left[500\left(e^{-0.4t} - e^{-0.5t}\right)\right]$$

$$= 250e^{-0.5t} - 200e^{-0.4t}$$

For maximum $F'(t) = 0$

$$\therefore 250e^{-0.5t} = 200e^{-0.4t}$$

$$\frac{5}{4} = \frac{e^{-0.4t}}{e^{-0.5t}}$$

$$= e^{0.1t}$$

$$\therefore 0.1t = \ln\left(\frac{5}{4}\right)$$

$$t = 10\ln\left(\frac{5}{4}\right)$$