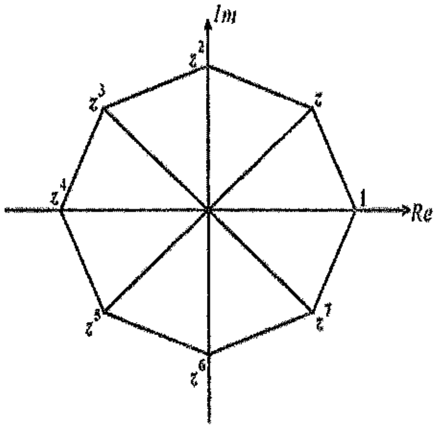
2017 Higher School Certificate Examination Paper Mathematics Extension 2

Section I

10 marks **Attempt Questions 1–10** **Allow about 15 minutes for this section**

1 The complex number z is chosen so that $1, z, \ldots, z^7$ form the vertices of the regular polygon shown.



Which polynomial equation has all of these complex numbers as roots?

(A) $x^7 - 1 = 0$ (B) $x^7 + 1 = 0$
(C) $x^8 - 1 = 0$ (D) $x^8 + 1 = 0$

2 Suppose l is a line and S is a point NOT on l.

The point P moves so that the distance from P to S is half the perpendicular distance from P to l.

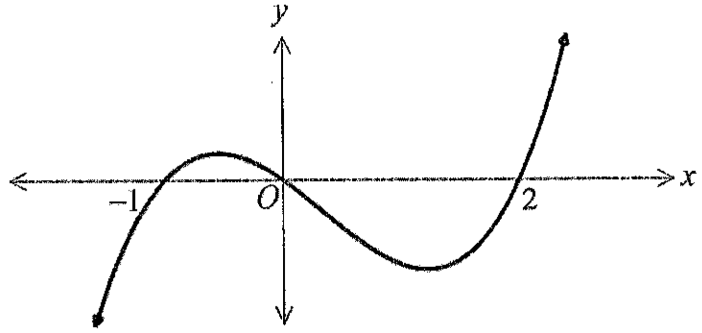
Which conic best describes the locus of P?

(A) A circle (B) An ellipse
(C) A parabola (D) A hyperbola

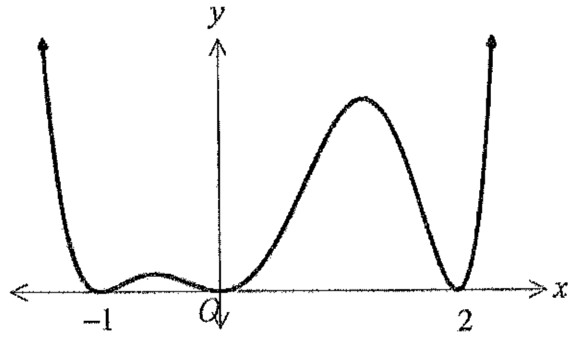
3 Which complex number lies in the region $2 < |z - 1| < 3$?

(A) $1 + \sqrt{3}i$ (B) $1 + 3i$
(C) $2 + i$ (D) $3 - i$

4 The graph of the function $y = f(x)$ is shown.



A second graph is obtained from the function $y = f(x)$.



Which equation best represents the second graph?

(A) $y = \sqrt{f(x)}$ (B) $y^2 = f(x)$

(C) $y = [f(x)]^2$ (D) $y = f(x^2)$

5 The polynomial $p(x) = x^3 - 2x + 2$ has roots α, β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

(A) -10 (B) -6

(C) -2 (D) 0

6 It is given that $z = 2 + i$ is a root of $z^3 + az^2 - 7z + 15 = 0$, where a is a real number.

What is the value of a?

(A) -1 (B) 1

(C) 7 (D) -7

7 It is given that $f(x)$ is a non-zero even function and $g(x)$ is a non-zero odd function.

Which expression is equal to $\int_{-a}^{a} f(x)+g(x)\,dx$?

(A) $2\int_0^a f(x)\,dx$

(B) $2\int_0^a g(x)\,dx$

(C) $\int_{-a}^{a} g(x)dx$

(D) $2\int_0^a f(x)+g(x)\,dx$?

8 Suppose that $f(x)$ is a non-zero odd function.

Which of the functions below is also odd?

(A) $f(x^2)\cos x$

(B) $f(f(x))$

(C) $f(x^3)\sin x$

(D) $f(x^2)-f(x)$

9 A particle is travelling on the circle with equation $x^2+y^2=16$.

It is given that $\frac{dx}{dt}=y$.

Which statement about the motion of the particle is true?

(A) $\frac{dy}{dt}=x$ and the particle travels clockwise

(B) $\frac{dy}{dt}=x$ and the particle travels anticlockwise

(C) $\frac{dy}{dt}=-x$ and the particle travels clockwise

(D) $\frac{dy}{dt}=-x$ and the particle travels anticlockwise

10 Suppose $f(x)$ is a differentiable function such that $\frac{f(a)+f(b)}{2}\geq f\left(\frac{a+b}{2}\right)$, for all a and b.

Which statement is always true?

(A) $\int_0^1 f(x)\,dx \geq \frac{f(0)+f(1)}{2}$

(B) $\int_0^1 f(x)\,dx \leq \frac{f(0)+f(1)}{2}$

(C) $f'\left(\frac{1}{2}\right)\geq 0$

(D) $f'\left(\frac{1}{2}\right)\leq 0$

Section II

90 marks Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

In Questions 11–16, responses should include relevant mathematical reasoning and/or calculations.

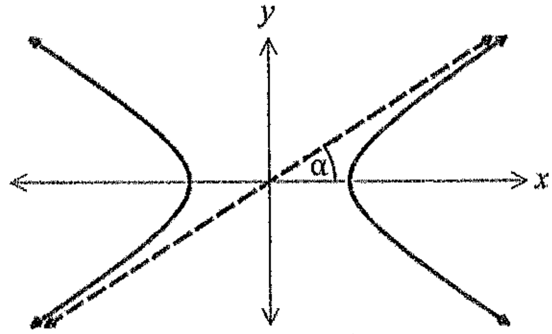
Question 11 (15 marks)

(a) Let $z = 1 - \sqrt{3}i$ and $w = 1 + i$.

(i) Find the exact value of the argument of z. **1**

(ii) Find the exact value of the argument of $\dfrac{z}{w}$. **2**

(b) An asymptote to the hyperbola $\dfrac{x^2}{12} - \dfrac{y^2}{4} = 1$ makes an angle α with the positive x-axis, as shown. **2**

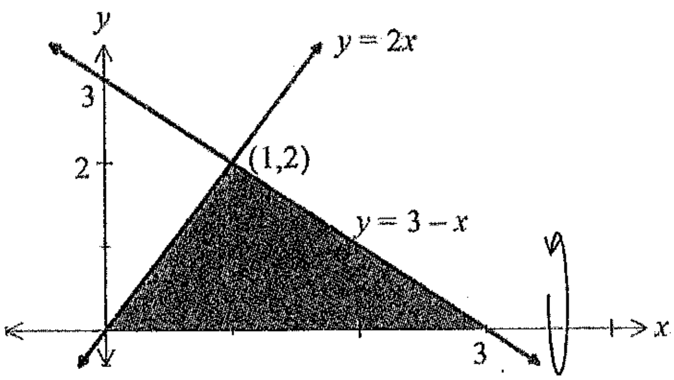


Find the value of α.

(c) Sketch the region in the Argand diagram where $-\dfrac{\pi}{4} \le \arg z \le 0$ and $|z - 1 + i| \le 1$. **2**

(d) Using the substitution $t = \tan\dfrac{\theta}{2}$, or otherwise, evaluate $\displaystyle\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}\,d\theta$. **3**

(e) The region bounded by the lines $y = 3 - x$, $y = 2x$ and the x-axis is rotated about the x-axis. **2**



Use the method of cylindrical shells to find an integral whose value is the volume of the solid of revolution formed. Do NOT evaluate the integral.

(f) Using the substitution $x=\sin^2\theta$, or otherwise, evaluate $\int_0^{\frac{1}{2}}\sqrt{\frac{x}{1-x}}\,dx$. **3**

Question 12 (15 marks)

(a) Consider the function $f(x)=\dfrac{e^x-1}{e^x+1}$.

(i) Show that $f(x)$ is increasing for all x. **1**

(ii) Show that $f(x)$ is an odd function. **1**

(iii) Describe the behaviour of $f(x)$ for large positive values of x. **1**

(iv) Hence sketch the graph of $f(x)=\dfrac{e^x-1}{e^x+1}$. **1**

(v) Hence, or otherwise, sketch the graph of $y=\dfrac{1}{f(x)}$. **1**

(b) Solve the quadratic equation $z^2+(2+3i)z+(1+3i)=0$, giving your answers in the form $a+bi$, where a and b are real numbers. **3**

(c) Find $\int x\tan^{-1}x\,dx$. **3**

(d) Let $P(x)$ be a polynomial.

(i) Given that $(x-\alpha)^2$ is a factor of $P(x)$ show that $P(\alpha)=P'(\alpha)=0$. **2**

(ii) Given that the polynomial $P(x)=x^4-3x^3+x^2+4$ has a factor $(x-\alpha)^2$, find the value of α. **2**

Question 13 (15 marks)

(a) Show that $\dfrac{r+s}{2}\geq\sqrt{rs}$ for $r\geq 0$ and $s\geq 0$. **1**

(b) Let a, b and c be real numbers. Suppose that $P(x)=x^4+ax^3+bx^2+cx+1$ has roots $\alpha,\dfrac{1}{\alpha},\beta,\dfrac{1}{\beta}$, where $\alpha>0$ and $\beta>0$.

(i) Prove that $a=c$. **2**

(ii) Using the inequality in part (a), show that $b\geq 6$. . **2**

(c) A particle is projected upwards from ground level with initial velocity $\frac{1}{2}\sqrt{\frac{g}{k}}$ ms^{-1}, where g is the acceleration due to gravity and k is a positive constant. The particle moves through the air with speed v ms^{-1} and experiences a resistive force. **4**

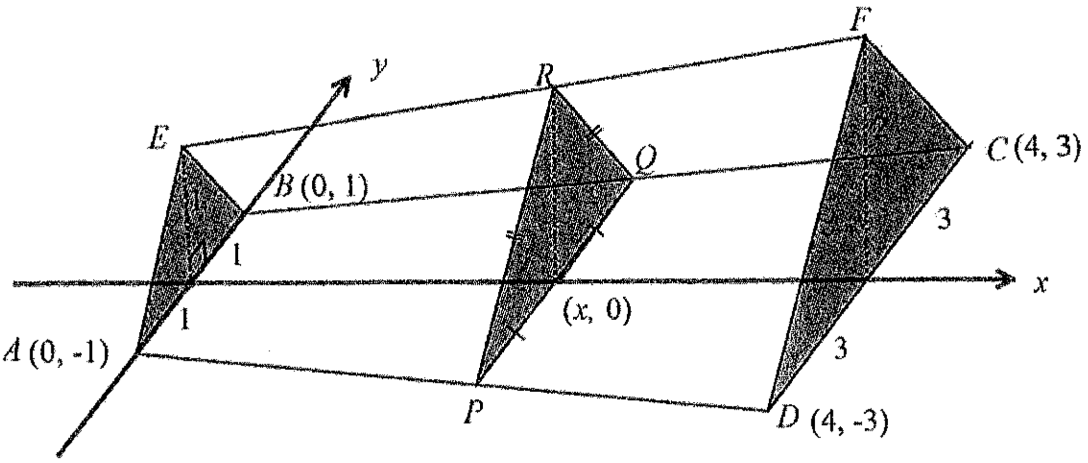
The acceleration of the particle is given by $\ddot{x} = -g - kv^2$ ms^{-2}. Do NOT prove this.

The particle reaches a maximum height, H, before returning to the ground.

Using $\ddot{x} = v\frac{dv}{dx}$, or otherwise, show that $H = \frac{1}{2k}\log_e\left(\frac{5}{4}\right)$ metres.

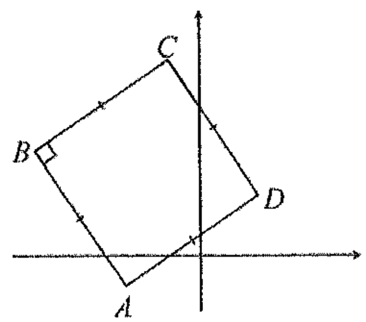
(d) The trapezium whose vertices are $A(0,-1), B(0,1), C(4,3)$ and $D(4,-3)$ forms the base of a solid. **4**

Each cross-section perpendicular to the $x-$axis is an isosceles triangle. The height of the isosceles triangle ABE with base AB, is 1. The height of the isosceles triangle DCF with base DC, is 2. The cross-section through the point $(x, 0)$ is the isosceles triangle PQR, where R lies on the line EF, as shown in the diagram.



Find the volume of the solid.

(e) The points A, B, C and D on the Argand diagram represent the complex numbers a, b, c and d respectively. The points form a square as shown on the diagram. **2**



By using vectors, or otherwise, show that $c = (1 + i)d - ia$.

Question 14 (15 marks)

(a) It is given that $x^4+4=\left(x^2+2x+2\right)\left(x^2-2x+2\right)$.

(i) Find A and B so that $\dfrac{16}{x^4+4}=\dfrac{A+2x}{x^2+2x+2}+\dfrac{B-2x}{x^2-2x+2}$. **1**

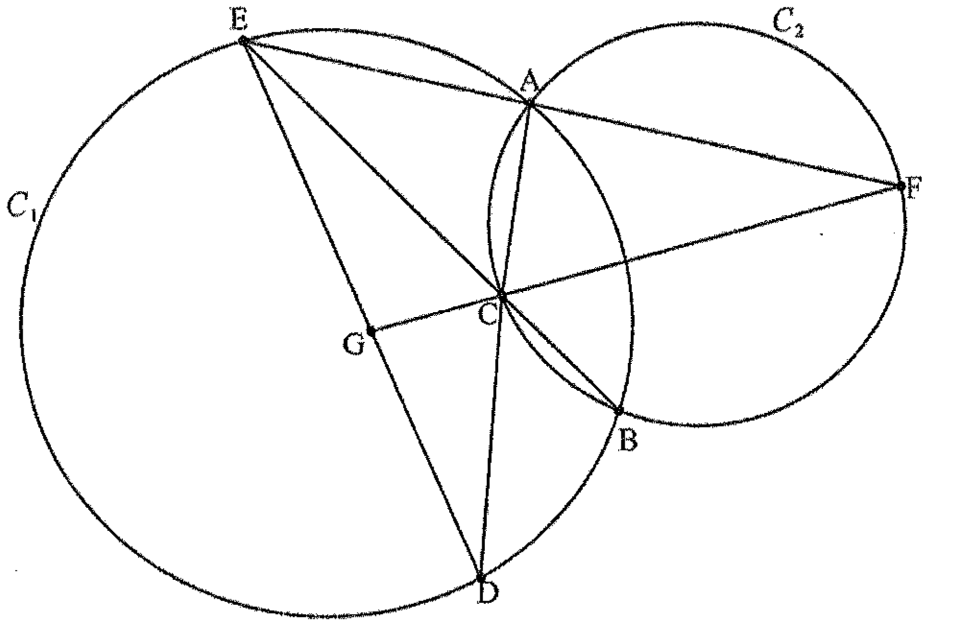
(ii) Hence, or otherwise, show that for any real number m, **2**

$$\int_0^m \frac{16}{x^4+4}\,dx=\ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1).$$

(iii) Find the limiting value as $m\to\infty$ of $\displaystyle\int_0^m \frac{16}{x^4+4}\,dx$. **1**

(b) Two circles C_1 and C_2, intersect at the points A and B. Point C is chosen on the arc AB of C_2 as shown in the diagram.

The line segment AC produced meets C_1 at D.
The line segment BC produced meets C_1 at E.
The line segment EA produced meets C_2 at F.
The line segment FC produced meets the line segment ED at G.



(i) State why $\angle EAD=\angle EBD$. **1**

(ii) Show that $\angle EDA=\angle AFC$. **1**

(iii) Hence, or otherwise, show that B, C, G and D are concyclic points. **3**

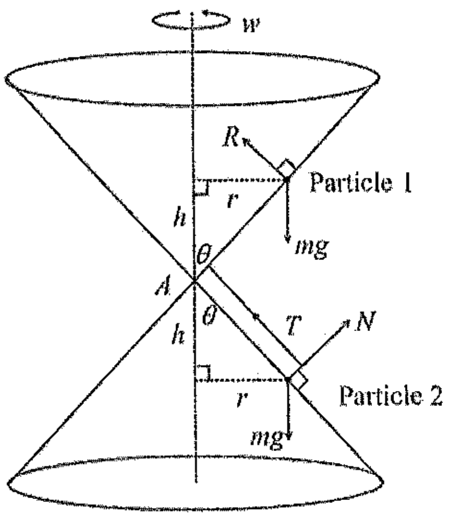
(c) A smooth double cone with semi-vertical angle $\theta < \frac{\pi}{2}$ is rotating about its axis with constant angular velocity w.

Two particles, each of mass m, are sitting on the cone as it rotates, as shown in the diagram.

Particle 1 is inside the cone at vertical distance h above the apex, A, and moves in a horizontal circle of radius r.

Particle 2 is attached to the apex A by a light inextensible string so that it sits on the cone at vertical distance h below the apex. Particle 2 also moves in a horizontal circle of radius r.

The acceleration due to gravity is g.



(i) The normal reaction force on Particle 1 is R. **2**

By resolving R into vertical and horizontal components, or otherwise, show that $w^2 = \frac{gh}{r^2}$.

(ii) The normal reaction force on Particle 2 is N and the tension in the string is T. **2**

By considering horizontal and vertical forces, or otherwise, show that

$$N = mg\left(\sin\theta - \frac{h}{r}\cos\theta\right)$$

(iii) Show that $\theta \geq \frac{\pi}{4}$. **2**

Question 15 (15 marks)

(a) Let $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$, for $n = 0, 1, 2, \ldots$.

(i) Find the value of I_1. **1**

(ii) Using integration by parts, or otherwise, show that for $n \geq 2$ **3**

$$I_n = \left(\frac{n-1}{n+2}\right) I_{n-2}.$$

(iii) Find the value of I_5. **1**

(b) Consider the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, for $x \geq 0$ and $y \geq 0$, where a is a positive constant.

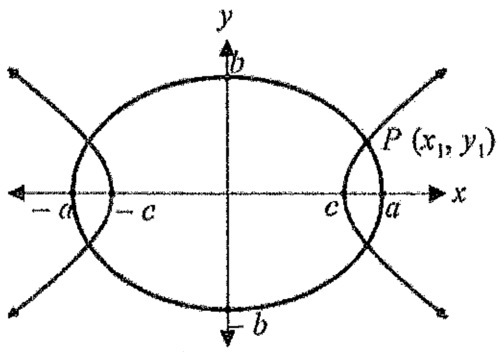
(i) Show that the equation of the tangent to the curve at the point $P(c, d)$ is given by $y\sqrt{c} + x\sqrt{d} = d\sqrt{c} + c\sqrt{d}$. **2**

(ii) The tangent to the curve at the point P meets the x and y axes at A and B respectively. Show that $OA + OB = a$, where O is the origin. **3**

(c) The ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, has eccentricity e.

The hyperbola with equation $\frac{x^2}{c^2} - \frac{y^2}{d^2} = 1$, has eccentricity E.

The value of c is chosen so that the hyperbola and the ellipse meet at $P(x_1, y_1)$, as shown in the diagram.



(i) Show that $\frac{x_1^2}{y_1^2} = \frac{a^2c^2}{(a^2 - c^2)} \times \frac{(b^2 + d^2)}{b^2d^2}$ **2**

(ii) If the two conics have the same foci, show that their tangents at P are perpendicular. **3**

Question 16 (15 marks)

(a) Let $\alpha = \cos\theta + i\sin\theta$, where $0 < \theta < 2\pi$.

(i) Show that $a^k + a^{-k} = 2\cos k\theta$, for any integer k. **1**

Let $C = \alpha^{-n} + \ldots + \alpha^{-1} + 1 + \alpha + \ldots + \alpha^n$, where n is a positive integer.

(ii) By summing the series, prove that $C = \dfrac{\alpha^n + \alpha^{-n} - \left(\alpha^{n+1} + \alpha^{-(n+1)}\right)}{(1-\alpha)(1-\overline{\alpha})}$. **3**

(iii) Deduce, from parts (i) and (ii), that **2**

$$1 + 2(\cos\theta + \cos 2\theta + \ldots + \cos n\theta) = \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}$$

(iv) Show that $\cos\dfrac{\pi}{n} + \cos\dfrac{2\pi}{n} + \ldots + \cos\dfrac{n\pi}{n}$ is independent of n. **1**

(b) The hyperbola with equation **2**

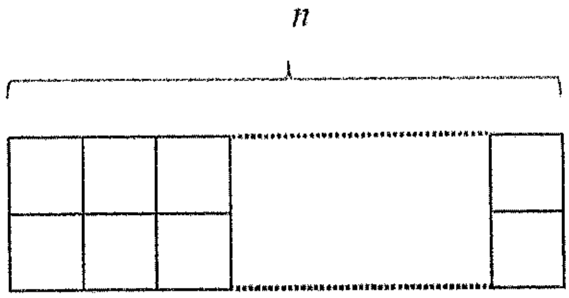
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has eccentricity 2.

The distance from one of the foci to one of the vertices is 1.

What are the possible values of a?

(c) A 2 by n grid is made up of two rows of n square tiles, as shown.



The tiles of the 2 by n grid are to be painted so that tiles sharing an edge are painted using different colours. There are x different colours available, where $x \geq 2$.

It is NOT necessary to use all the colours.

Consider the case of the 2 by 2 grid with tiles labelled A, B, C and D, as shown.

A	C
B	D

There are $x(x-1)$ ways to choose colours for the first column containing tiles A and B. Do NOT prove this.

(i) Assume the colours for tiles A and B have been chosen. There are two cases to consider when choosing colours for the second column. Either tile C is the same colour as tile B, or tile C is a different colour from tile B. **2**

By considering these two cases, show that the number of ways of choosing colours for the second column is

$$x^2 - 3x + 3.$$

(ii) Prove by mathematical induction that the number of ways in which the 2 by n grid can be painted is $x(x-1)(x^2-3x+3)^{n-1}$, for $n \geq 1$. **2**

(iii) In how many ways can a 2 by 5 grid be painted if 3 colours are available and each colour must now be used at least once? **2**

End of paper

2017 Higher School Certificate Solutions Mathematics Extension 2

Section I

Multiple Choice Summary

1 C	**2** B	**3** D	**4** C	**5** B
6 A	**7** A	**8** B	**9** C	**10** B

Multiple Choice Solutions

1. **C**

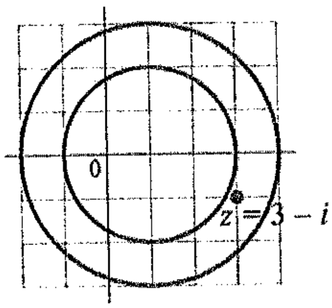
It has 8 roots so it is either (C) or (D). Must be (C) because 1 is a root.

2. **B**

$PS = ePM$ but $e = \frac{1}{2}$, so it must be an ellipse since $0 < e < 1$.

3. **D**

Note that $|(3-i)-1| = |2-i| = \sqrt{4+1} = \sqrt{5}$ which is between 2 and 3



$3-i$ is the only point that lies between the two circles.

4. **C**

All single roots became double roots for $y = [f(x)]^2$ and all original x-intercepts of $y = f(x)$ become minimum turning points. $y = [f(x)]^2$ is always non-negative.

5. **B**

$$\begin{aligned}
\alpha^3 &= 2\alpha - 2 \\
\beta^3 &= 2\beta - 2 \\
\gamma^3 &= 2\gamma - 2 \\
\alpha^3 + \beta^3 + \gamma^3 &= 2(\alpha + \beta + \gamma) - 6 \\
&= 2(0) - 6 \\
&= -6
\end{aligned}$$

6. **A**

Since the coefficients are real one other root is $2-i$ Let γ be the third root.

Method 1

Using the product:

$$\alpha\beta\gamma=-15$$
$$(2+i)(2-i)\gamma=-15$$
$$(4-i^2)\gamma=-15$$
$$5\gamma=-15$$
$$\gamma=-3$$

Using the sum:

$$\alpha+\beta+\gamma=-a$$
$$(2+i)+(2-i)-3=-a$$
$$4-3=-a$$
$$a=-1$$

Method 2

Same as Method 1 to find $\gamma=-3$.

Then substitute $\gamma=-3$ into original equation and solve for a:

$$(-3)^3+a(-3)^2-7(-3)+15=0$$
$$-27+9a+21+15=0$$
$$9a=-9$$

and hence $a=-1$.

7. **A**

$$\int[f(x)+g(x)]\,dx=\int f(x)\,dx+\int g(x)\,dx$$

$\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$ since $f(x)$ is even

$\int_{-a}^{a} g(x)\,dx=0$ since $g(x)$ is odd

$$\therefore \int_{-a}^{a}[f(x)+g(x)]\,dx=2\int_0^a f(x)\,dx$$

8. **B**

Now $f(-x)=-f(x)$

so, $f\left(f(-x)\right)=f\left(-f(x)\right)$

$=-f\left(f(x)\right)$ since $f(x)$ is an odd function.

Thus $f\left(f(x)\right)$ is itself odd.

Note: In a multiple choice situation, we can substitute any odd function and see which solution works. Eg. $f(x)=x^5$ then $f(f(x))=(x^5)^5=x^{25}$ which is an odd function.

9. C

Method 1

$x^2 + y^2 = 16$

Differentiating implicitly w.r.t. t:

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2xy + 2y\frac{dy}{dt} = 0$

$2y\left(x + \frac{dy}{dt}\right) = 0$

Thus $\frac{dy}{dt} = -x$ (typically $y \neq 0$)

Method 2

$x^2 + y^2 = 16$

Differentiating implicitly w.r.t. x:

$2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-x}{y}$

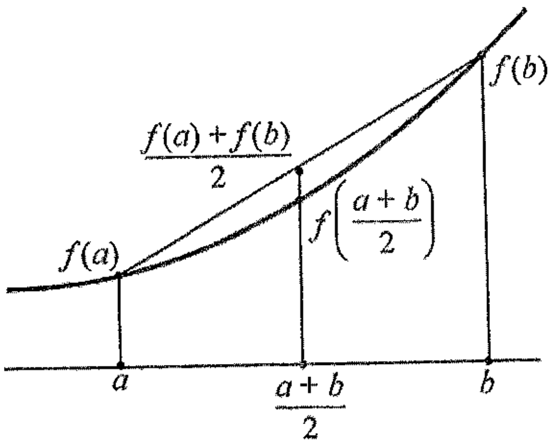
Now $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

So $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{-x}{y} \times y = -x$

Now choose any known point on the original circle, e.g. $x = 0, y = 4$, where $\frac{dx}{dt} = 4$ which is a clockwise direction of motion since it is moving in a positive x-direction.

10. B

Consider the diagram below. We note that $\frac{f(a)+f(b)}{2}$ is the mid-point of the chord and $f\left(\frac{a+b}{2}\right)$ is the value of the function $f(x)$ at the mid-point of the interval from $x = a$ to $x = b$. Since $\frac{f(a)+f(b)}{2} \geq f\left(\frac{a+b}{2}\right)$, we know that $f(x)$ must be concave-up.

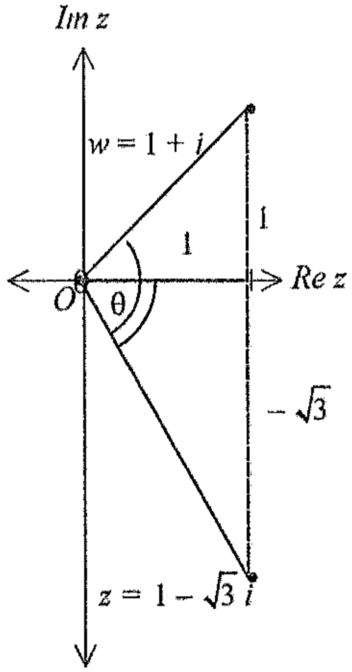


Letting $a = 0$ and $b = 1$, $\int_0^1 f(x)\,dx$ represents the area under the graph, which is less than the area of the trapezium. Therefore the answer must be B.

Section II

Question 11

(a) (i)



From the diagram

$\theta = \arg(z)$

$= \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$

$= -\frac{\pi}{3}$

(ii)

$\arg(w) = \tan^{-1}\left(\frac{1}{1}\right)$

$= \frac{\pi}{4}$

$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$

$= -\frac{\pi}{3} - \frac{\pi}{4}$

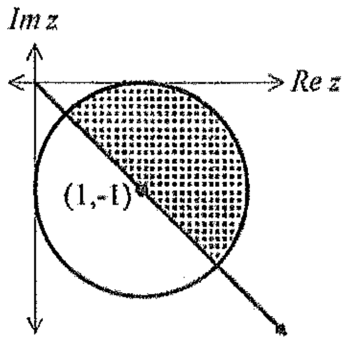
$= -\frac{7\pi}{12}$

(b) $a = \sqrt{12} = 2\sqrt{3}, b = 2$

The equations of the asymptotes are $y = \pm\frac{b}{a}x = \pm\frac{2}{2\sqrt{3}}x = \pm\frac{1}{\sqrt{3}}x$

Now since the gradient is positive then $\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$

(c)



$|z-(1-i)| \le 1$ represents the area inside the circle, centred on $1-i$ with radius 1.

$-\frac{\pi}{4} \le \arg z \le 0$ represents the area above the line with argument $-\frac{\pi}{4}$, below the real axis.

(d) **Method 1**

Since $t = \tan\frac{\theta}{2}$

$$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$$

$$dt = \frac{1}{2}(1+t^2)\,d\theta$$

$$\therefore d\theta = \frac{2dt}{1+t^2}$$

$$\theta = 0 \;\Rightarrow\; t = \tan\frac{0}{2} = 0$$

$$\theta = \frac{2\pi}{3} \;\Rightarrow\; t = \tan\frac{\frac{2\pi}{3}}{2}\tan\frac{\pi}{3} = \sqrt{3}$$

Now:

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}\,d\theta = \int_0^{\sqrt{3}} \frac{1}{\left(\frac{1+t^2}{1+t^2}\right)+\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int_0^{\sqrt{3}} \frac{1}{\left(\frac{2}{1+t^2}\right)}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int_0^{\sqrt{3}} 1\,dt$$

$$= [t]_0^{\sqrt{3}}$$

$$= \sqrt{3}$$

Method 2

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta}\,d\theta = \int_0^{\frac{2\pi}{3}} \frac{1-\cos\theta}{\sin^2\theta}\,d\theta$$

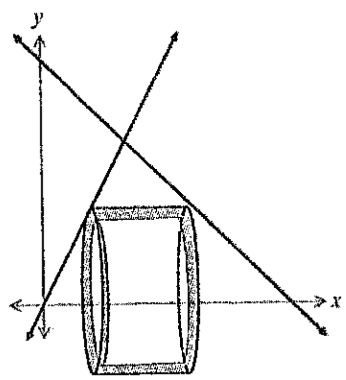
$$= \int_0^{\frac{2\pi}{3}} \csc^2\theta - \csc\theta\cot\theta\,d\theta$$

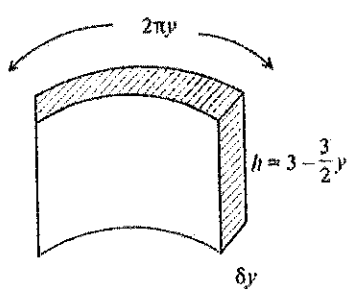
$$= \left[-\cot\theta + \csc\theta\right]_0^{\frac{2\pi}{3}}$$

$$= \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}$$

$$= \sqrt{3}$$

(e) First, construct a horizontal strip of thickness δy and rotate that about the x-axis. The cylindrical shell will be oriented such that the axis of symmetry is horizontal. The height of the cylinder is the difference of the x coordinates, which is $h = x_2 - x_1 = (3-y) - \frac{y}{2} = 3 - \frac{3}{2}y$. The radius is just the y coordinate of any point on the strip, which is just y.





Hence, the volume can be given by

$$\begin{aligned}\delta V &= 2\pi y(h)\delta y \\ &= 2\pi y\left(3-\frac{3}{2}y\right)\delta y \\ V &= 2\pi\int_0^2 y\left(3-\frac{3}{2}y\right)dy \\ &= 3\pi\int_0^2 y(2-y)\,dy\end{aligned}$$

(f) When $x=0,\ \theta=0$ and when $x=\frac{1}{2},\ \theta=\frac{\pi}{4}$

Compute the differential term $dx = 2\sin\theta\cos\theta\, d\theta$.

$$\begin{aligned}I &= \int_0^{\frac{\pi}{4}}\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\times 2\sin\theta\cos\theta\, d\theta \\ &= \int_0^{\frac{\pi}{4}}\frac{\sin\theta}{\cos\theta}\times 2\sin\theta\cos\theta\, d\theta \\ &= 2\int_0^{\frac{\pi}{4}}\sin^2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{4}}1-\cos 2\theta\, d\theta \\ &= \left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4}-\frac{1}{2}\end{aligned}$$

Question 12

(a) $f(x)=\dfrac{e^x-1}{e^x+1}$

(i)
$$\begin{aligned}f'(x) &= \frac{e^x\left(e^x+1\right)-e^x\left(e^x-1\right)}{\left(e^x+1\right)^2} \\ &= \frac{e^{2x}+e^x-e^{2x}+e^x}{\left(e^x+1\right)^2} \\ &= \frac{2e^x}{\left(e^x+1\right)^2} > 0 \text{ for all } x \text{ since } e^x>0 \text{ and } \left(e^x+1\right)^2>0\end{aligned}$$

(ii) $f(-x)=\frac{e^{-x}-1}{e^{-x}+1}\times\left(\frac{e^x}{e^x}\right)$

$=\frac{1-e^x}{1+e^x}$

$=-\frac{e^x-1}{e^x+1}$

$=-f(x)$

Therefore $f(x)$ is an odd function.

(iii) **Method 1**

$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{e^x-1}{e^x+1}$

$=\lim_{x\to\infty}\frac{1-e^{-x}}{1+e^{-x}}$

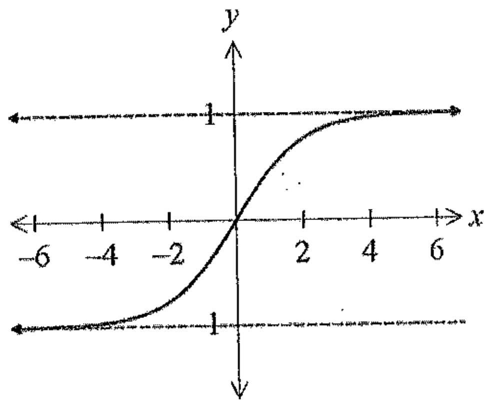
$=1$

Method 2

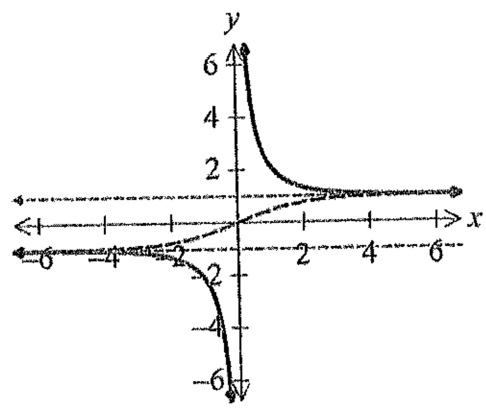
Now, $f(x)=\frac{e^x+1-2}{e^x+1}=1-\frac{2}{e^x+1}$.

Thus, as $x\to\infty, f(x)\to 1^-$, since $\frac{2}{e^x+1}\to 0^+$.

(iv)



(v)



(b) **Method 1**

$z^2+(2+3i)z+(1+3i)=0$

$z^2+(2+3i)z+\frac{(2+3i)^2}{4}=\frac{(2+3i)^2}{4}-(1+3i)$

$\left(z+\frac{2+3i}{2}\right)^2=\frac{4+12i-9}{4}-1-3i$

$\left(z+\frac{2+3i}{2}\right)^2=-\frac{9}{4}$

$z+\frac{2+3i}{2}=\pm\frac{3}{2}i$

$z=\frac{3}{2}i-\frac{2+3i}{2}, -\frac{3}{2}i-\frac{2+3i}{2}$

$=-1, -1-3i$

Method 2

$z=\frac{-(2+3i)\pm\sqrt{(2+3i)^2-4(1+3i)}}{2}$

$=\frac{-2-3i\pm\sqrt{4+12i+9i^2-4-12i}}{2}$

$=\frac{-2-3i\pm\sqrt{-9}}{2}$

$=\frac{-2-3i\pm 3i}{2}$

$=-1, -1-3i$

(c) Use integration by parts:

$u = \tan^{-1} x$ $\qquad dv = x\,dx$

$du = \dfrac{dx}{1+x^2}$ $\qquad v = \dfrac{1}{2}x^2$

$$\begin{aligned} I &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C \\ &= \frac{1}{2}\left(x^2 \tan^{-1}(x) - x + \tan^{-1}(x)\right) + C \end{aligned}$$

(d) (i) $P(x) = (x-\alpha)^2 A(x)$ where $A(x)$ is a polynomial of degree two less than degree of $P(x)$.

Hence $P(\alpha) = 0$

$$\begin{aligned} P'(x) &= 2(x-\alpha)A(x) + (x-\alpha)^2 A'(x) \\ &= (x-\alpha)\left[2A(x) + (x-\alpha)A'(x)\right] \end{aligned}$$

Hence $P'(\alpha) = 0$

So $P(\alpha) = P'(\alpha) = 0$

(ii) $$\begin{aligned} P(x) &= x^4 - 3x^3 + x^2 + 4 \\ P'(x) &= 4x^3 - 9x^2 + 2x \\ &= x\left(4x^2 - 9x + 2\right) \\ &= x(4x-1)(x-2) \end{aligned}$$

Solving this, and using (i), the possible multiple roots are $0, \frac{1}{4}, 2$.

Since $P(x)$ is monic, try $x = 2$.

$$\begin{aligned} P(2) &= 2^4 - 3(2)^3 + 2^2 + 4 \\ &= 16 - 24 + 4 + 4 \\ &= 0 \end{aligned}$$

Hence $\alpha = 2$

Question 13

(a) $\left(\sqrt{r}-\sqrt{s}\right)^2 \geq 0$

$r-2\sqrt{rs}+s \geq 0$

$r+s \geq 2\sqrt{rs}$

$\dfrac{r+s}{2} \geq \sqrt{rs}$

(b) (i) $P(x)=x^4+ax^3+bx^2+cx+d$

Sum of roots 1 at a time $=\alpha+\dfrac{1}{\alpha}+\beta+\dfrac{1}{\beta}=-a$

Sum of roots 3 at a time $=\alpha\dfrac{1}{\alpha}\beta+\beta\dfrac{1}{\beta}\alpha+\alpha\dfrac{1}{\beta}\dfrac{1}{\alpha}+\beta\dfrac{1}{\beta}\dfrac{1}{\alpha}=-c$

$=\beta+\alpha+\dfrac{1}{\beta}+\dfrac{1}{\alpha}=-c$

$\therefore a=c$ as required.

(ii) Note that using part (a), the sum of any positive number and its reciprocal $\geq$ 2:

$r+s \geq 2\sqrt{rs}$

$\dfrac{a}{b}+\dfrac{b}{a} \geq 2\sqrt{\dfrac{a}{b}\cdot\dfrac{b}{a}}=2$

Sum of roots 2 at a time

$b=\alpha\dfrac{1}{\alpha}+\beta\dfrac{1}{\beta}+\alpha\dfrac{1}{\beta}+\beta\dfrac{1}{\alpha}+\alpha\beta+\dfrac{1}{\beta}\dfrac{1}{\alpha}$

$b=1+1+\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right)+\left(\alpha\beta+\dfrac{1}{\alpha\beta}\right)$

$\geq 2+2\sqrt{\dfrac{\alpha}{\beta}\dfrac{\beta}{\alpha}}+2\sqrt{\alpha\beta\dfrac{1}{\alpha\beta}}$ using (a)

$\geq 2+2\sqrt{1}+2\sqrt{1}$

≥ 6 as required

(c) $$v\frac{dv}{dx} = -\left(g + kv^2\right)$$

$$\int \frac{v\,dv}{g + kv^2} = -\int dx$$

$$\int \frac{2kv\,dv}{g + kv^2} = -2k\int dx$$

$$\ln(g + kv^2) = -2kx + C$$

But at $x = 0, v = \frac{1}{2}\sqrt{\frac{g}{k}}$ so,

$$\ln\left(g + k\left(\frac{1}{2}\sqrt{\frac{g}{k}}\right)^2\right) = -2k(0) + C$$

$$\ln\left(g + k\frac{g}{4k}\right) = C$$

$$C = \ln\left(\frac{5g}{4}\right)$$

Therefore, $\ln(g + kv^2) = -2kx + \ln\left(\frac{5g}{4}\right)$

$$2kx = \ln\left(\frac{5g}{4}\right) - \ln(g + kv^2)$$

$$x = \frac{1}{2k}\ln\left(\frac{5g}{4(g + kv^2)}\right)$$

Now at maximum height $v = 0$ and $x = H$

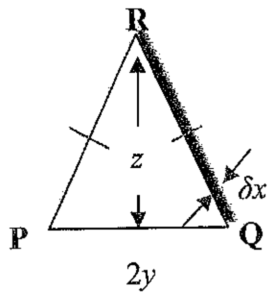
Thus, $H = \frac{1}{2k}\ln\left(\frac{5g}{4(g + k(0)^2)}\right)$

$= \frac{1}{2k}\ln\left(\frac{5}{4}\right)$ as required.

(d) Let the third (vertical axis) be z. Then the equations of BC and EF by inspection are:

$$BC: y = \frac{1}{2}x + 1 \quad \text{and} \quad EF: z = \frac{1}{4}x + 1$$

Consider an element of volume of a cross-section of the solid, δV through $(x, 0)$.



$$\delta V = \frac{2yz}{2}\delta x = yz\delta x$$

Hence

$$\delta V = \frac{1}{2}(x+2)\frac{1}{4}(x+4)\delta x$$

$$= \frac{x^2+6x+8}{8}\delta x$$

$$\text{Thus } V = \int_0^4 \frac{x^2+6x+8}{8}dx = \frac{1}{8}\left[\frac{x^3}{3}+3x^2+8x\right]_0^4 = \frac{304}{24} = 12\frac{2}{3}$$

(e) **Method 1**

$\overrightarrow{DC} = \overrightarrow{DA}$ rotated clockwise 90^0

$c - d = (a-d)\times -i$

$c = -ai + di + d$

$= -ai + d(i+1)$

$= (1+i)d - ia$

Method 2

$\overrightarrow{AD} = d - a = \overrightarrow{BC}$

$\overrightarrow{AB} = i\,\overrightarrow{AD} = i(d-a) = \overrightarrow{DC}$

but $\overrightarrow{DC} = c - d$

so $c - d = id - ia$

$\therefore c = d + id - ia = (1+i)d - ia$

Question 14

(a) (i) Cross multiply and equate numerators.

$$16 \equiv (A+2x)(x^2-2x+2)+(B-2x)(x^2+2x+2)$$

Equate the constant term.

$$2A+2B=16$$
$$A+B=8$$

By symmetry, A and B must both be 4.

Or, alternatively substitute a value for x into the equation above and solve simultaneously,
Eg. let $x=1$, then

$$16=(A+2)(1-2+2)+(B-2)(1+2+2)$$
$$16=A+2+5(B-2)$$
$$A+5B=24$$

Solving simultaneously with $A+B=8$ gives $A=4$ and $B=4$.

(ii)

$$\int_0^m \frac{dx}{x^4+16}=\int_0^m \frac{4+2x}{x^2+2x+2}+\frac{4-2x}{x^2-2x+2}dx$$
$$=\int_0^m \frac{2+2x}{x^2+2x+2}+\frac{2}{x^2+2x+2}dx-\int_0^m \frac{2x-4}{x^2-2x+2}dx$$
$$=\int_0^m \frac{2x+2}{x^2+2x+2}+\frac{2}{1+(x+1)^2}dx-\int_0^m \frac{2x-2}{x^2-2x+2}-\frac{2}{1+(x-1)^2}dx$$
$$=\left[\ln(x^2+2x+2)+2\tan^{-1}(x+1)\right]_0^m-\left[\ln(x^2-2x+2)-2\tan^{-1}(x-1)\right]_0^m$$
$$=\ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1)+\ln 2-\ln 2+2\left(\frac{\pi}{4}\right)+2\left(-\frac{\pi}{4}\right)$$
$$=\ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1) \text{ as required.}$$

(iii) Each inverse tan term becomes $2\times\frac{\pi}{2}=\pi$ and the inside of the log function approaches 1, hence the entire log function approaches zero.

Therefore the limit is 2π.

(b) (i) They are both angles subtended from the same arc ED in circle C_1.

(ii) We know that $\angle EDA=\angle EBA$ for the same reason as (i), but with arc EA. Now looking at circle C_2, we can see that arc AC subtends both $\angle ABC$ and $\angle AFC$.
Hence $\angle EDA=\angle EBA=\angle CBA=\angle AFC$.

(iii) Let $\angle EAD = \theta$, which means, from part (i) $\angle EBD = \theta$.

from (ii), we can deduce that $AFDG$ is a cyclic quadrilateral (angles $\angle AFG (= \angle AFC)$ and $\angle ADG (= \angle EDA)$ subtended by the same chord (AG) are equal). Hence, $\angle FAD = \angle FGD$ (angles subtended by the same arc are equal).

But $\angle FAD = \pi - \theta$ (angles on the straight line EF) therefore $\angle FGD = \pi - \theta$. So now we have $\angle CGD + \angle CBD = \pi - \theta + \theta = \pi$, showing that $BCGD$ is a cyclic quadrilateral (opposite angles are supplementary).

(c) (i) Resolving Horizontally: $R\cos\theta = mr\omega^2$

Resolving Vertically: $R\sin\theta = mg$

Dividing both: $\dfrac{R\sin\theta}{R\cos\theta} = \dfrac{mg}{mr\omega^2}$

$$\tan\theta = \frac{g}{r\omega^2}$$

From the diagram $\tan\theta = \dfrac{r}{h}$, so by equating these:

$$\frac{r}{h} = \frac{g}{r\omega^2}$$

$$\text{Thus } \omega^2 = \frac{gh}{r^2}.$$

(ii) Resolving Horizontally: $T\sin\theta - N\cos\theta = mr\omega^2$ ...(1)

Resolving Vertically: $T\cos\theta + N\sin\theta = mg$ ...(2)

Using the elimination method $(2)\times\sin\theta - (1)\times\cos\theta$:

$$N\left(\sin^2\theta + \cos^2\theta\right) = mg\sin\theta - mr\omega^2\cos\theta$$

$$N = mg\left(\sin\theta - \frac{r\omega^2}{g}\cos\theta\right)$$

$$= mg\left(\sin\theta - \frac{h}{r}\cos\theta\right) \text{ ... from part (i)}$$

(iii) For their to be a reaction force on Particle 2, we need $N \ge 0$.

$mg\left(\sin\theta - \dfrac{h}{r}\cos\theta\right) \ge 0$ and since $mg \ne 0$, $\sin\theta \ge \dfrac{h}{r}\cos\theta$

$$\text{Thus, } \tan\theta \ge \frac{h}{r}$$

$$\ge \frac{1}{\tan\theta} \quad \text{... since } \tan\theta = \frac{h}{r}$$

$$\tan^2\theta \ge 1$$

$$\tan\theta \ge 1 \qquad \text{Hence } \theta \ge \frac{\pi}{4}$$

Question 15

(a) (i)

$$I_1 = \int_0^1 x\sqrt{1-x^2}$$

$$= -\frac{1}{3}\left[\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= -\frac{1}{3}\left[\left(1-1^2\right)^{\frac{3}{2}} - \left(1-0^2\right)^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}\times -1$$

$$= \frac{1}{3}$$

(ii) $u = x^{n-1}$ $\qquad dv = x\sqrt{1-x^2}\,dx$

$du = (n-1)x^{n-2}$ $\qquad v = -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}$

Using integration by parts:

$$I_n = \left[-\frac{1}{3}x^{n-1}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1 + \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)^{\frac{3}{2}}dx$$

$$= [0-0] + \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)\sqrt{\left(1-x^2\right)}\,dx$$

$$= \frac{n-1}{3}\int_0^1 \left(x^{n-2}-x^n\right)\sqrt{\left(1-x^2\right)}\,dx$$

$$= \frac{n-1}{3}\int_0^1 x^{n-2}\sqrt{\left(1-x^2\right)} - x^n\sqrt{\left(1-x^2\right)}\,dx$$

$$= \frac{n-1}{3}\left(I_{n-2} - I_n\right).$$

Rearrange to make I_n the subject.

$$3I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$(n+2)I_n = (n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n+2}I_{n-2}$$

(iii) From part (i), we know that $I_1 = \frac{1}{3}$. Using the recurrence we get:

$$\begin{aligned} I_3 &= \frac{n-1}{n+2} I_1 \\ &= \frac{3-1}{3+2} I_1 \\ &= \frac{2}{5} I_1 \\ &= \frac{2}{5} \times \frac{1}{3} \\ &= \frac{2}{15}. \end{aligned}$$

Repeating this process:

$$\begin{aligned} I_5 &= \frac{5-1}{5+2} I_3 \\ &= \frac{4}{7} I_3 \\ &= \frac{4}{7} \times \frac{2}{15} \\ &= \frac{8}{105}. \end{aligned}$$

(b) (i) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and implicitly differentiating w.r.t. x,

$$\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} y^{-\frac{1}{2}} \frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{2\sqrt{y}}{2\sqrt{x}} \\ &= -\sqrt{\frac{y}{x}} \end{aligned}$$

Hence, the gradient of the tangent at P is $m = -\sqrt{\frac{d}{c}}$.

Use the point-gradient formula to find the equation of the tangent:

$$y - d = -\sqrt{\frac{d}{c}}(x - c)$$

$$y\sqrt{c} - d\sqrt{c} = -x\sqrt{d} + c\sqrt{d}$$

$$x\sqrt{d} + y\sqrt{c} = d\sqrt{c} + c\sqrt{d}$$

(ii) The intercept of the tangent is:

$$OA=\frac{d\sqrt{c}+c\sqrt{d}}{\sqrt{d}}$$
$$=\sqrt{cd}+c$$

Similarly, the y intercept is $OB=\sqrt{cd}+d$. Now we can add them:

$$\begin{aligned}OA+OB&=2\sqrt{cd}+c+d\\&=\left(\sqrt{c}+\sqrt{d}\right)^2\\&=\left(\sqrt{a}\right)^2 \qquad \ldots\left(\text{since } P \text{ lies on } \sqrt{x}+\sqrt{y}=\sqrt{a}\right)\\&=a\end{aligned}$$

(c) (i) P lies on both the ellipse and hyperbola. Hence, we have two equations.

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$
$$\frac{x_1^2}{c^2}-\frac{y_1^2}{d^2}=1$$

Thus:

$$\begin{aligned}\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}&=\frac{x_1^2}{c^2}-\frac{y_1^2}{d^2}\\x_1^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)&=-y_1^2\left(\frac{1}{b^2}+\frac{1}{d^2}\right)\\x_1^2\left(\frac{c^2-a^2}{a^2c^2}\right)&=-y_1^2\left(\frac{b^2+d^2}{b^2d^2}\right)\\\frac{x_1^2}{y_1^2}&=-\frac{\left(b^2+d^2\right)}{b^2d^2}\div\frac{\left(c^2-a^2\right)}{a^2c^2}\\\frac{x_1^2}{y_1^2}&=-\frac{\left(b^2+d^2\right)}{b^2d^2}\times\frac{a^2c^2}{\left(c^2-a^2\right)}\\&=\frac{a^2c^2}{\left(a^2-c^2\right)}\times\frac{\left(b^2+d^2\right)}{b^2d^2}\end{aligned}$$

(ii) The tangent has the following gradient at P

$$\frac{2x_1}{a^2}+\frac{2y_1}{b^2}\left(\frac{dy}{dx}\right)=0$$
$$\frac{2y_1}{b^2}\left(\frac{dy}{dx}\right)=-\frac{2x_1}{a^2}$$
$$\frac{dy}{dx}=-\frac{b^2x_1}{a^2y_1}$$

Similarly, the hyperbola has gradient $\frac{dy}{dx}=\frac{d^2x_1}{c^2y_1}$

Multiply their gradients to get $m_1m_2=-\frac{b^2x_1}{a^2y_1}\times\frac{d^2x_1}{c^2y_1}=-\frac{b^2d^2}{a^2c^2}\times\frac{x_1{}^2}{y_1{}^2}$

If the two conics have the same foci, then $ae_E=ce_H$ and hence $a^2-b^2=c^2+d^2$, since $a^2e^2=a^2-b^2$ in an ellipse and $c^2e^2=c^2+d^2$ in the hyperbola.

Substitute into (i)

$$m_1m_2=-\frac{b^2d^2}{a^2c^2}\times\frac{a^2c^2}{\left(a^2-c^2\right)}\times\frac{\left(b^2+d^2\right)}{b^2d^2}$$
$$=-\frac{b^2+d^2}{a^2-c^2} \qquad \ldots\left(\text{since } a^2-b^2=c^2+d^2\right)$$

$=-1$. Therefore the two gradients are perpendicular.

Question 16

(a) (i) Standard result.

$$\alpha^k+\alpha^{-k}=\text{cis}(k\theta)+\text{cis}(-k\theta)$$
$$=2\,\text{Re}\left(\text{cis}(k\theta)\right)$$
$$=2\cos(k\theta)$$

(ii) The series is a sum of a GP. Note that $\frac{1}{\alpha}=\text{cis}(-\theta)=\bar{\alpha}$.

$$C=\frac{\alpha^{-n}\left(1-\alpha^{2n+1}\right)}{1-\alpha}=\frac{\alpha^{-n}\left(1-\alpha^{2n+1}\right)\left(1-\alpha^{-1}\right)}{(1-\alpha)(1-\bar{\alpha})}$$
$$=\frac{\left(\alpha^{-n}-\alpha^{n+1}\right)\left(1-\alpha^{-1}\right)}{(1-\alpha)(1-\bar{\alpha})}$$
$$=\frac{\alpha^{-n}-\alpha^{-n-1}-\alpha^{n+1}+\alpha^{n}}{(1-\alpha)(1-\bar{\alpha})}$$
$$=\frac{\alpha^{n}+\alpha^{-n}-\left(\alpha^{n+1}+\alpha^{-(n+1)}\right)}{(1-\alpha)(1-\bar{\alpha})}$$

(iii) Substitute in (i) repeatedly to the series definition of C.

$$C = 1 + \left(\alpha + \alpha^{-1}\right) + \left(\alpha^2 + \alpha^{-2}\right) + \ldots + \left(\alpha^n + \alpha^{-n}\right)$$
$$= 1 + 2\cos\theta + 2\cos 2\theta + \ldots + 2\cos n\theta$$

Now substitute into the closed form for C in (ii).

$$C = \frac{\left(\alpha^n + \alpha^{-n}\right) - \left(\alpha^{n+1} + \alpha^{-(n+1)}\right)}{(1-\alpha)(1-\bar{\alpha})}$$
$$= \frac{2\cos n\theta - 2\cos(n+1)\theta}{1 - (\alpha + \bar{\alpha}) + \alpha\bar{\alpha}}$$
$$= \frac{2\cos n\theta - 2\cos(n+1)\theta}{1 - (2\operatorname{Re}\alpha) + 1}$$
$$= \frac{2\cos n\theta - 2\cos(n+1)\theta}{2 - 2\cos\theta}$$
$$= \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}$$

Equate the two expressions and we acquire the result.

(iv) Re-arrange the result in (iii) to get

$$1 + 2\cos\theta + 2\cos 2\theta + \ldots + 2\cos n\theta = \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}$$
$$\cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2}\left(\frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta} - 1\right)$$

Let $\theta = \dfrac{\pi}{n}$.

$$\cos\left(\frac{\pi}{n}\right) + \cos\left(\frac{2\pi}{n}\right) + \ldots + \cos\left(\frac{n\pi}{n}\right) = \frac{1}{2}\left(\frac{\cos\left(\frac{n\pi}{n}\right) - \cos\left(\frac{n+1}{n}\pi\right)}{1 - \cos\left(\frac{\pi}{n}\right)} - 1\right)$$
$$= \frac{1}{2}\left(\frac{-1 - \cos\left(\pi + \frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)} - 1\right)$$
$$= \frac{1}{2}\left(\frac{-1 + \cos\left(\frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)} - 1\right)$$
$$= -1$$

And this is independent of n.

(b) We are given that $e=2$ and $|a|e\pm|a|=1$, so solving for a yields

$$|a|e\pm|a|=1$$
$$|a|(e\pm 1)=1$$
$$a=\pm\frac{1}{3} \text{ or } a=\pm 1$$

(c) (i) Case #1: Tile C is the same colour as B.

A	B
B	?

Tile D can be anything except the colour of B, so therefore there are $(x-1)$ possibilities.

Case #2: Tile C is a different colour to B.

A	?
B	?

Tile C can be anything except A or B's colour, so it has $(x-2)$ possibilities. Similarly, tile D can be anything except B or C's colour, so it also has $(x-2)$ possibilities. Hence, the total is $(x-2)^2+(x-1)=x^2-3x+3$ possibilities.

(ii) Base Case: $n=1$

We have a 2×1 grid, which has $x(x-1)$ possibilities (given), which matches the formula for $n=1$.

Inductive Hypothesis: $n=k$
A $2\times k$ grid will have $x(x-1)(x^2-3x+3)^{k-1}$ possible colourings.
Inductive Step: $n=k \Rightarrow n=k+1$
Consider a $2\times(k+1)$ grid, which is basically a slight extension of the grid in the inductive hypothesis. Use the result from (i) here.
Before the extension, we had $x(x-1)(x^2-3x+3)^{k-1}$ possible colourings. By throwing on the extra two tiles, we multiply in the result by the number of possibilities in (i), which is (x^2-3x+3). Hence, we have

$$x(x-1)(x^2-3x+3)^{k-1}\times(x^2-3x+3)=x(x-1)(x^2-3x+3)^{k}$$

and the induction is complete.

(iii) Substitute $x = 3$ into (ii) to get 486. But this will include a number of 'bad' cases because the equation in (ii) was constructed under the assumption that not all colours have to be used.

However, in this question, we must use all colours at least once. So we will manually subtract the 'bad' case where we use only two colours (note that we can't use one colour only).

Choose 3 out of 2 colours to be used in $\binom{3}{2}$ ways. But once we pick the two colours, we have only two possible colourings that zig-zag across the grid in an alternating manner.

Hence, there are $2\times\binom{3}{2} = 6$ bad cases.

Therefore, the total number of colourings is $486 - 6 = 480$ possible colourings.

End of Mathematics Extension 2 solutions