KAMBALA

Student Number:

September 2016

Preliminary
Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Board approved calculators may be used.
- Writing using black or blue pen.
- All necessary working should be shown in Question 8 - 10.
- Write your student number and/or name at the top of every page.

Total marks – 52

Section I – Pages 2 – 4
7 marks
Attempt Questions 1 – 7
Allow about 10 minutes for this section

Section II – Pages 5 – 7
45 marks
Attempt Questions 8 - 10
Allow about 1 hour and 20 minutes for this section

This paper MUST NOT be removed from the examination room

Section I

7 Marks
Attempt Questions 1 – 7
Allow about 10 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 7.

Marks

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1. $A(-5,8)$ and $B(1,2)$ are two points. What are the coordinates of the point P that divides AB externally in the ratio 3:1? **1**

(A) $(-1,4)$

(B) $(-2,5)$

(C) $(5,-2)$

(D) $(4,-1)$

2. On a test there are 10 true/false questions and 5 multiple choice questions with four possible answers. How many different choices are there for answering the 15 questions? **1**

(A) 27396

(B) 218400

(C) 400032

(D) 1048576

3. If $y = x(2x+1)^4$ which of the following is an expression for $\frac{dy}{dx}$? **1**

(A) $8x(2x+1)^3$

(B) $(6x+1)(2x+1)^3$

(C) $(10x+1)(2x+1)^3$

(D) $(2x+1)^4$

4. What is the size of the acute angle between the lines $3x - y = 0$ and $x + 2y = 0$, correct to the nearest degree? **1**

(A) 45°

(B) 54°

(C) 82°

(D) 79°

5. Given $P(x)$ is an even polynomial and $Q(x)$ is an odd polynomial which is a divisor of $P(x)$ then $\frac{P(x)}{Q(x)} + P(x)Q(x)$ is **1**

(A) an odd polynomial

(B) an even polynomial

(C) neither odd nor even

(D) not a polynomial

6. How many numbers greater than 4000 can be formed with the numbers 2, 3, 4, 5 and 6 if each digit is used only once? **1**

(A) 72

(B) 172

(C) 120

(D) 192

7. Which of the following is an expression for $\frac{\sin(A+B)}{\cos(A-B)}$? **1**

(A) $\frac{\tan A - \tan B}{1 - \tan A \tan B}$

(B) $\frac{\tan A - \tan B}{1 + \tan A \tan B}$

(C) $\frac{\tan A + \tan B}{1 + \tan A \tan B}$

(D) $\frac{\tan A + \tan B}{1 - \tan A \tan B}$

Section II

45 Marks
Attempt Questions 11 – 14
Allow about 1 hour and 20 minutes for this section

Answer the questions on the writing paper provided.
Start each question on a new page.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 8 (15 marks) **Start a new page.** **Marks**

(a) Find $\lim_{x \to 3} \frac{x-3}{x^2-9}$. **2**

(b) Simplify $\frac{3^{2-x} \times 81}{9^{x+3}}$. **2**

(c) (i) Write down the exact values of sin 60° and sin 45°. **1**

(ii) Hence find the exact value of sin (15°). **2**

(d) Consider the polynomial $P(x) = 2x^3 + 3x^2 - 11x - 6$.

(i) Show that $(x - 2)$ is a factor of $P(x)$. **1**

(ii) Solve the equation $P(x) = 0$. **2**

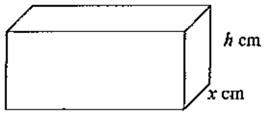
(e) Solve the inequality $\frac{2x}{x-1} > 1$. **2**

(f) Use the principle of mathematical induction to prove that $n^3 + 2n$ is divisible by 3 for all positive integers n. **3**

Question 9 (15 marks) **Start a new page.** **Marks**

(a) Solve the equation $\frac{n!}{(n-2)!} = 20$. **2**

(b) Radioactive waste is to be disposed of in fully enclosed lead boxes of volume 200 cm^3. The base of the box has dimensions in the ratio 2:1. Let x be the width of the box and h the height in cm.



(i) Show that $h = \frac{100}{x^2}$. **1**

(ii) Show that the surface area of the box is given by $A(x) = 4x^2 + \frac{600}{x}$ cm^2. **1**

(iii) Find the value of x so that the surface area of the box is a minimum. **2**

(c) Find the value(s) of m such that the acute angle between the lines $y = mx$ and $y = 2x$ is 45°. **2**

(d) Solve the equation $\sin 2x + \cos x = 0$ for $0° \le x \le 360°$. **2**

(e) (i) Sketch the graph of the function $y = 2\sin\frac{1}{2}x - 1$ for $0 \le x \le 360°$. **2**

(ii) Hence determine the number of solutions to the equation $2\sin\left(\frac{1}{2}x\right) = 1$ for $0° \le x \le 360°$. **1**

(f) Solve the equation $3\tan(x - 30) = \sqrt{3}$ for $0° \le x \le 360°$. **2**

Question 10 (15 marks) **Start a new page.** **Marks**

(a) The equation $2x^3 - 8x^2 + 1 = 0$ has roots α, β and γ. Find $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **2**

(b) Use the principle of mathematical induction to prove that **3**

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

(c) Find the number of ways in which the letters of the word CARBON can be arranged in a line

(i) without restriction **1**

(ii) so that the 2 vowels are next to each other, but the 4 consonants are not all next to each other. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to solve the equation $2 + \cos x - 2\sin x = 0$ for $0^0 \le x \le 360^0$, giving answers correct to the nearest degree. **3**

(e) Show that $\cos^2 2x + \frac{1}{2}\sin^2 2x = \cos^4 x + \sin^4 x$. **2**

(f) The polynomial $P(x)$ is such that $P(x) = (x^2 - 1)Q(x) + kx + 2$ for some polynomial $Q(x)$ and some constant k. If $(x - 1)$ is a factor of $P(x)$, find the remainder when $P(x)$ is divided by $(x + 1)$. **2**

End of Paper

Student Number: _

Section I

7 Marks
Attempt Questions 1 – 7
Allow about 10 minutes for this section

Select the alternative A, B, C or D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				X
2				X
3			X	
4			X	
5	X			
6				X
7			X	

(15) Well Done!!

Question 8

a) $\lim_{x \to 3} \frac{(x-3)}{(x+3)(x-3)}$ ✓

$\lim_{x \to 3} \frac{1}{x+3}$

$= \frac{1}{3+3} = \frac{1}{6}$ ✓ 2

b) $\frac{3^{2-x} \times 3^4}{(3^2)^{x+3}}$ ✓ $= \frac{3^{6-x}}{3^{2x+6}}$

$= 3^{(6-x)-(2x+6)}$

$= 3^{\cancel{6}-x-2x-\cancel{6}} = 3^{-3x}$ ✓ 2

c) i) $\sin 60^\circ = \frac{\sqrt{3}}{2}$ ✓ 1

$\sin 45^\circ = \frac{1}{\sqrt{2}}$

ii) $\sin 15^\circ = \sin(60^\circ - 45^\circ)$

$= \sin 60 \cos 45^\circ - \cos 60^\circ \sin 45^\circ$

$= \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right)$ ✓

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ ✓

2

d i) $P(x) = 2x^3 + 3x^2 - 11x - 6$

For $(x-2)$ to be a factor, $P(2) = 0$

$P(2) = 2(8) + 3(4) - 11(2) - 6$

$= 16 + 12 - 22 - 6$

$= 6$ ✓ 1

$\therefore (x-2)$ is a factor

ii) $\cancel{x-2 \,)\, 2x^3 + 3x^2 - 11x - 6}$

$\cancel{2x^3 - 4x^2}$

$\cancel{x^2 - 11x - 6}$

$\cancel{x^2 - 2x}$

$\cancel{-9x - 6}$

$$2x^2 + 7x + 3$$ ✓

$$x - 2 \,\overline{)\, 2x^3 + 3x^2 - 11x - 6}$$

$2x^3 - 4x^2$

$7x^2 - 11x - 6$

$7x^2 - 14x$

$3x - 6$

$3x - 6$

0

$P(x) = (x-2)(2x^2 + 7x + 3)$

$= (x-2)(2x^2 + x + 6x + 3)$

$= (x-2)(x(2x+1) + 3(2x+1))$

$= (x-2)(2x+1)(x+3)$

$(x-2)(2x+1)(x+3) = 0$

$x = 2, \; x = -\frac{1}{2}, \; x = -3$ ✓

2

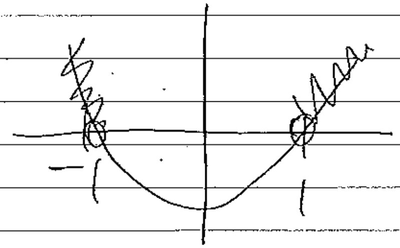
e) $\dfrac{2x}{x-1} > 1$ $\quad x \neq 1$

$2x(x-1) > (x-1)^2$ ✓

$2x(x-1) - (x-1)^2 > 0$

$(x-1)[2x - x + 1] > 0$

$(x-1)(x+1) > 0$



$x < -1, \; x > 1$ ✓ 2

f) $n^3 + 2n$ by 3

Prove true for $n = 1$

$1^3 + 2(1)$

$= 1 + 2$

$= 3$ which is divisible by 3

$\therefore$ True for $n = 1$ ✓

Assume true for $n = k$

$k^3 + 2k = 3p$ (where p is a positive integer)

$k^3 = 3p - 2k$

Required to prove true for $n=k+1$
$(k+1)^3 + 2(k+1) = 3q$ (where q is another integer)

LHS: $(k+1)(k^2+2k+1) + 2k+2$
$k^3 + 2k^2 + k + k^2 + 2k + 1 + 2k + 2$
$= k^3 + 3k^2 + 5k + 3$
$= (3p - 2k) + 3k^2 + 5k + 3$
$= 3p + 3k^2 + 3k + 3$
$= 3(p + k^2 + k + 1)$
$= 3q$ (since p is a positive integer and so is k and 1.)
$= RHS$

$\therefore$ true for $n = k+1$

$\therefore$ Since true for $n=1$ and $n=k+1$ when true for $n=k$ the result is true for ~~an~~ all positive integers.

3

Question 9 (10)

a) $\frac{n!}{(n-2)!} = 20$

$\frac{n!}{(n-2)(n-1)n!} = 20$

$\frac{1}{n^2-3n+2} = 20$

$1 = 20(n^2-3n+2)$

$1 = 20n^2 - 60n + 40$

$20n^2 - 60n + 39 = 0$

Using quadratic formula:

$n = \frac{60 \pm \sqrt{(-60)^2 - 4(20 \times 39)}}{2(20)}$

$= \frac{60 \pm \sqrt{480}}{40}$

$= \frac{60 \pm \sqrt{16}\sqrt{30}}{40}$

$= \frac{60 \pm 4\sqrt{30}}{40}$

$= \frac{15 \pm \sqrt{30}}{10}$

$0/2$

$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$
$= n(n-1) = 20$
$n^2 - n - 20 = 0$

Expand the numerator.

$n = -4$
$n = 5$
but $n > 0$
$\therefore \boxed{n = 5}$

b) i) volume $= 200cm^3$ Base $= 2x$

$h \times x \times 2x = 200$

$2x^2h = 200$

1/1 $h = \frac{200}{2x^2}$

$h = \frac{100}{x^2}$

ii) surface Area:

$2hx + 2x(x) + 2h(2x)$

$= \frac{200x}{x^2} + 4x^2 + 2\left(\frac{2 \times 100}{x^2}\right)$

1/1 $= \frac{200}{x} + 4x^2 + 2\left(\frac{200}{x^2}\right)$

$= 4x^2 + \frac{200+400}{x} = 4x^2 + \frac{600}{x} cm$

iii) $\frac{dA}{dx} = 8x + \left(\frac{-600}{x^2}\right)$

At a minimum, $\frac{dA}{dx} = 0$

$8x - \frac{600}{x^2} = 0$

$8x^3 - 600 = 0$

2/2 $8x^3 = 600$

$x^3 = 75$

$x = \sqrt[3]{75}$

$x = 4.217163327$ cm

$m_1 = m \quad m_2 = 2$

c) $\tan 45° = \left|\frac{m-2}{1+2m}\right|$

$1 = \frac{m-2}{1+2m}$

0/2 $1-2m = m-2$

$-3m = -3$

$m = 1$

$1+2m = m-2$

$m = -3$

$\frac{m-2}{1+m} = -1$

$m-2 = -m-1$

$2m = 1 \quad m = \frac{1}{2}$

d) $\sin 2x + \cos x = 0 \quad 0 \leq x \leq 360°$

$2\sin x \cos x + \cos x = 0$

$\cos x(2\sin x + 1) = 0$

$\cos x = 0$

$x = 90°, 270°$

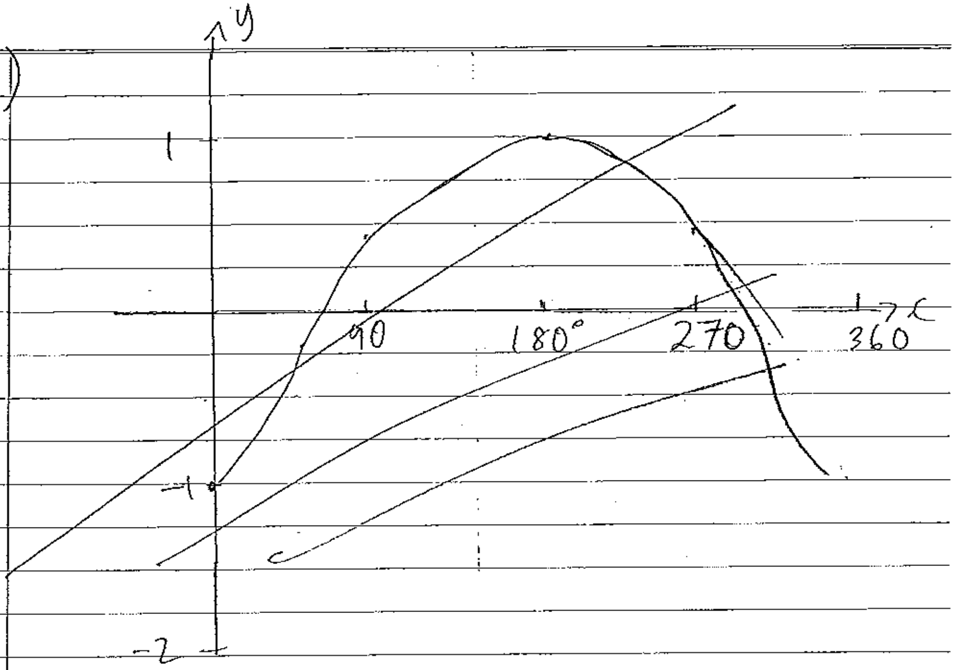
$2\sin x + 1 = 0$

$\sin x = -\frac{1}{2}$

$x = 210°, 330°$

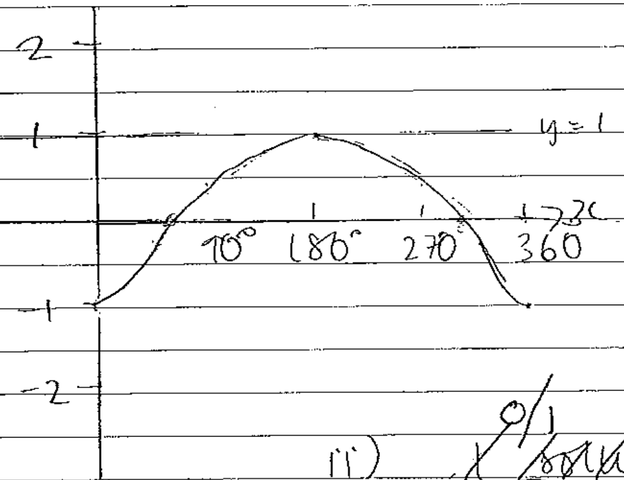
2/2 $\therefore x = 90°, 210°, 270°, 330°$

e(i)



ii)

2/2



iii) ~~1 solution~~ 0/1

~~x~~2 solutions

f) $3\tan(x-30) = \sqrt{3}$ $\quad 0 \leq x \leq 360$

$$\tan(x-30°) = \frac{\sqrt{3}}{3} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\tan(x-30°) = \frac{\cancel{3}}{\cancel{3}\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

Adjust domain: $-30° \leq x-30° \leq 330°$

$$x-30° = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$(x-30°) = 30°, 210°$$

2/2

$$x = (30°+30°), (210°+30°)$$

$$= 60°, 240°$$

Remember to check for 180°.

$$\tan(180-30) \neq \frac{1}{\sqrt{3}}$$

↳ with tan, always check for 180°

Question 10 (14)

a) $2x^3 - 8x^2 + 1 = 0$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\beta + \alpha\gamma}{\alpha\beta\gamma}$$

$$= \frac{\frac{c}{a}}{-\frac{d}{a}}$$

$2x^3 - 8x^2 + 0x + 1 = 0$

$\frac{c}{a} = \frac{0}{2} = 0$

$\frac{-d}{a} = \frac{-1}{2}$

2/2 $\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{0}{-\frac{1}{2}} = 0$

b) $$\sum_{r=1}^{n} (2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

LHS:
$(2-1)^2 + (4-1)^2 + (6-1)^2 + (2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Prove true for $r = 1$

LHS: $(2-1)^2 = 1$

RHS: $\frac{1(2-1)(2+1)}{3} = \frac{1(3)}{3} = 1$

$\therefore$ LHS = RHS

$\therefore$ True for $n = 1$

Assume true for $r = k$

$(2-1)^2 + (4-1)^2 + (6-1)^2 + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

Required to prove true for $r = k+1$

$(2-1)^2 + (4-1)^2 + (6-1)^2 + (2k-1)^2 + (2(k+1)-1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$

LHS:

$= \underbrace{(2-1)^2 + (4-1)^2 + (6-1)^2 + (2k-1)^2}_{\text{a assumption}} + (2k+1)^2$

$\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$

$(2k+1)\left[\frac{k(2k-1)}{3} + 2k+1\right]$

$= \frac{2k+1}{3}\left[2k^2 - k + 6k + 3\right]$

$= \frac{2k+1}{3}\left[2k^2 + 5k + 6\right]$

$= \frac{2k+1}{3}\left[2k^2 + 2k + 3k + 6\right]$

$= \frac{2k+1}{3}\left[2k(k+1) + 3(k+1)\right]$

$= \frac{2k+1}{3}(2k+3)(k+1)$

$= \frac{(k+1)(2k+1)(2k+3)}{3}$

3/3 = RHS

$\therefore$ True for $n = k+1$

$\therefore$ since true for $r = 1$, and $r = k+1$ when true for $r = k$, the result is true for $r \geq 1$

c i) CARBON

$= {}^6P_6 = 6! = 720$

ii) 3/3

$(3 \times 2!) \times 4! = 144$ ways

d) $2 + \cos x - 2\sin x = 0$ $\quad 0 \le x \le 360$

$2 + \frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2} = 0$

$= \frac{2+2t}{1+t^2} + \frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2}$

$= \frac{3-2t-t^2}{1-t^2}$

$= \frac{-t^2-2t+3}{1-t^2} = 0$

$-t^2-2t+3 = 0$

$t^2+2t-3 = 0$

$(t+3)(t-1) = 0$

$t \ne -3$ or $t = 1$ $\quad t > 0$

$\tan\frac{x}{2} = -3$ $\quad 0 \le \frac{x}{2} \le 180$

$\frac{x}{2} = \tan^{-1}(-3)$

$= 108°26'$ $\quad = 108°$

$\frac{x}{2} = 71°34'$

$x = 143°$

2/3

$\tan\frac{x}{2} = 1$

$\frac{x}{2} = 45°$

$\boxed{x = 90°}$

LHS:

e) $\cos^2 2x + \frac{1}{2}\sin^2 2x$

$(\cos^2 x - \sin^2 x)^2 + \frac{1}{2}(2\sin x\cos x)^2$

$= \cos^4 x - 2\cos^2 x\sin^2 x + \sin^4 x + \frac{1}{2}(4\sin^2 x\cos^2 x)$

$= \cos^4 x - \cancel{2\cos^2 x\sin^2 x} + \sin^4 x + \cancel{2\sin^2 x\cos^2 x}$

2/2 $= \cos^4 x + \sin^4 x$

$=$ RHS $\therefore$ proven

f) $P(1) = 0$

$P(1) = (1-1)Q(1) + k + 2 = 0$

$k + 2 = 0$

$\boxed{k = -2}$

$P(-1) = (1-1)Q(-1) - k + 2$

$= -k + 2$

$= -(-2) + 2$

$= 2 + 2$

$= 4$

2/2

$\therefore$ Remainder when divided by $(x+1)$ is 4.