NSW EDUCATION STANDARDS AUTHORITY

2017 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 Which polynomial is a factor of $x^3 - 5x^2 + 11x - 10$?

A. $x - 2$

B. $x + 2$

C. $11x - 10$

D. $x^2 - 5x + 11$

2 It is given that $\log_a 8 = 1.893$, correct to 3 decimal places.

What is the value of $\log_a 4$, correct to 2 decimal places?

A. 0.95

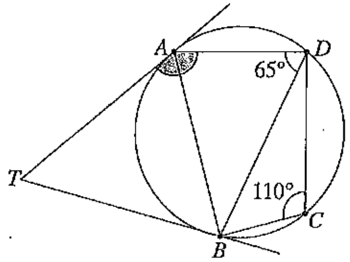
B. 1.26

C. 1.53

D. 2.84

3 The points A, B, C and D lie on a circle and the tangents at A and B meet at T, as shown in the diagram.

The angles BDA and BCD are $65°$ and $110°$ respectively.



What is the value of $\angle TAD$?

A. $130°$

B. $135°$

C. $155°$

D. $175°$

4 What is the value of $\tan\alpha$ when the expression $2\sin x - \cos x$ is written in the form $\sqrt{5}\sin(x-\alpha)$?

A. -2

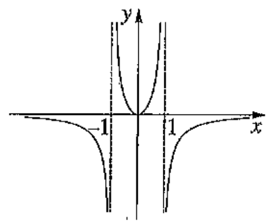
B. $-\dfrac{1}{2}$

C. $\dfrac{1}{2}$

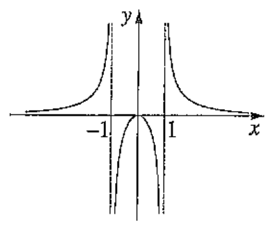
D. 2

5 Which graph best represents the function $y = \dfrac{2x^2}{1-x^2}$?

A.



B.



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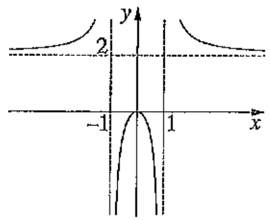

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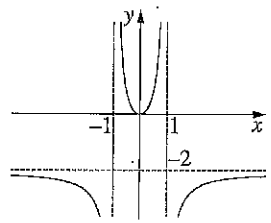


6 The point $P\left(\dfrac{2}{p}, \dfrac{1}{p^2}\right)$, where $p \neq 0$, lies on the parabola $x^2 = 4y$.

What is the equation of the normal at P?

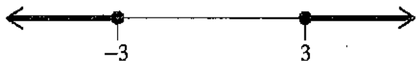
A. $py - x = -p$

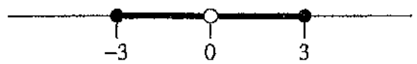
B. $p^2y + px = -1$

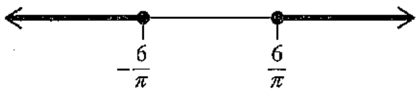
C. $p^2y - p^3x = 1 - 2p^2$

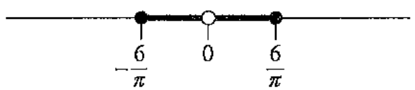
D. $p^2y + p^3x = 1 + 2p^2$

7 Which diagram represents the domain of the function $f(x)=\sin^{-1}\left(\dfrac{3}{x}\right)$?

A. 

B. 

C. 

D. 

8 A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of $5\ \text{cm s}^{-1}$.

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. $25\pi\ \text{cm}^2\,\text{s}^{-1}$

B. $30\pi\ \text{cm}^2\,\text{s}^{-1}$

C. $150\pi\ \text{cm}^2\,\text{s}^{-1}$

D. $225\pi\ \text{cm}^2\,\text{s}^{-1}$

9 When expanded, which expression has a non-zero constant term?

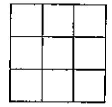
A. $\left(x+\dfrac{1}{x^2}\right)^7$

B. $\left(x^2+\dfrac{1}{x^3}\right)^7$

C. $\left(x^3+\dfrac{1}{x^4}\right)^7$

D. $\left(x^4+\dfrac{1}{x^5}\right)^7$

10 Three squares are chosen at random from the 3×3 grid below, and a cross is placed in each chosen square.



What is the probability that all three crosses lie in the same row, column or diagonal?

A. $\dfrac{1}{28}$

B. $\dfrac{2}{21}$

C. $\dfrac{1}{3}$

D. $\dfrac{8}{9}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval from $A(-4,-4)$ to $B(1,6)$ internally in the ratio $2:3$. **1**

Find the x-coordinate of P.

(b) Differentiate $\tan^{-1}(x^3)$. **2**

(c) Solve $\dfrac{2x}{x+1} > 1$. **3**

(d) Sketch the graph of the function $y = 2\cos^{-1}x$. **2**

(e) Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$. **3**

(f) Find $\displaystyle\int \sin^2 x \cos x\,dx$. **1**

Question 11 continues on page 8

(g) The probability that a particular type of seedling produces red flowers is $\frac{1}{5}$. Eight of these seedlings are planted.

(i) Write an expression for the probability that exactly three of the eight seedlings produce red flowers. **1**

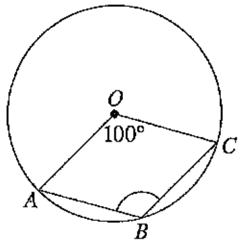
(ii) Write an expression for the probability that none of the eight seedlings produces red flowers. **1**

(iii) Write an expression for the probability that at least one of the eight seedlings produces red flowers. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle AOC$ is $100°$. **2**



NOT TO SCALE

Find the size of $\angle ABC$, giving reasons.

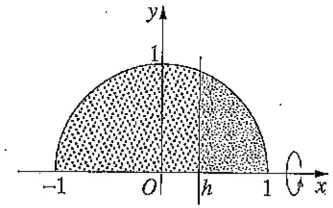
(b) (i) Carefully sketch the graphs of $y = |x+1|$ and $y = 3 - |x-2|$ on the same axes, showing all intercepts. **3**

(ii) Using the graphs from part (i), or otherwise, find the range of values of x for which **1**

$$|x+1| + |x-2| = 3.$$

Question 12 continues on page 10

(c) The region enclosed by the semicircle $y = \sqrt{1-x^2}$ and the x-axis is to be divided into two pieces by the line $x = h$, where $0 \le h < 1$.



The two pieces are rotated about the x-axis to form solids of revolution. The value of h is chosen so that the volumes of the solids are in the ratio $2:1$.

(i) Show that h satisfies the equation $3h^3 - 9h + 2 = 0$. **3**

(ii) Given $h_1 = 0$ as the first approximation for h, use one application of Newton's method to find a second approximation for h. **1**

(d) At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$. **3**

Find the acceleration of the particle as a function of x.

(e) Evaluate $\lim\limits_{x \to 0} \dfrac{1-\cos 2x}{x^2}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the x-axis in simple harmonic motion centred at the origin. **3**

When $x = 2$ the velocity of the particle is 4.

When $x = 5$ the velocity of the particle is 3.

Find the period of the motion.

(b) Let n be a positive EVEN integer.

(i) Show that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$. **2**

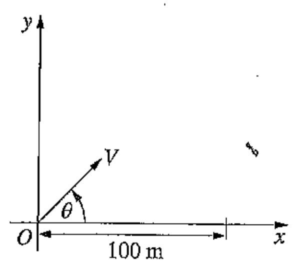
(ii) Hence show that **1**

$$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right].$$

(iii) Hence show that $\binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n} = n2^{n-3}$. **2**

Question 13 continues on page 12

(c) A golfer hits a golf ball with initial speed $V\ \mathrm{m\,s^{-1}}$ at an angle θ to the horizontal. The golf ball is hit from one side of a lake and must have a horizontal range of 100 m or more to avoid landing in the lake.



Neglecting the effects of air resistance, the equations describing the motion of the ball are

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time in seconds after the ball is hit and g is the acceleration due to gravity in $\mathrm{m\,s^{-2}}$. Do NOT prove these equations.

(i) Show that the horizontal range of the golf ball is $\dfrac{V^2 \sin 2\theta}{g}$ metres. **2**

(ii) Show that if $V^2 < 100g$ then the horizontal range of the ball is less than 100 m. **1**

It is now given that $V^2 = 200g$ and that the horizontal range of the ball is 100 m or more.

(iii) Show that $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$. **2**

(iv) Find the greatest height the ball can achieve. **2**

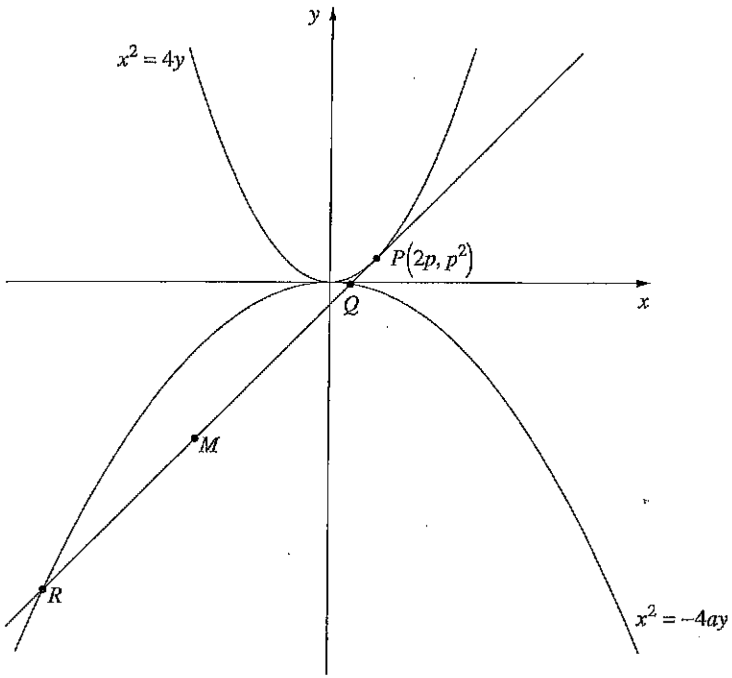
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that $8^{2n+1} + 6^{2n-1}$ is divisible by 7, for any integer $n \geq 1$. **3**

(b) Let $P\left(2p, p^2\right)$ be a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P meets the parabola $x^2 = -4ay,\ a > 0,$ at Q and R. Let M be the midpoint of QR.



(i) Show that the x coordinates of R and Q satisfy **2**

$$x^2 + 4apx - 4ap^2 = 0.$$

(ii) Show that the coordinates of M are $\left(-2ap, -p^2(2a+1)\right)$. **2**

(iii) Find the value of a so that the point M always lies on the parabola $x^2 = -4y$. **2**

Question 14 continues on page 14

(c) The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

(i) By differentiating the product $F(t)e^{0.4t}$ show that **2**

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. **2**

(iii) The concentration of the drug increases to a maximum. **2**

For what value of t does this maximum occur?

End of paper

1 ' Solutions

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Mathematics Extension 1

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Section I – 10 marks (pages 2–6)

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1260

Section I

10 marks
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Use the multiple-choice answer sheet for Questions 1–10.

1 Which polynomial is a factor of $x^3 - 5x^2 + 11x - 10$?

$8 - 20 + 22 - 10$

A. $x - 2$

B. $x + 2$

C. $11x - 10$

D. $x^2 - 5x + 11$

2 It is given that $\log_a 8 = 1.893$, correct to 3 decimal places.

What is the value of $\log_a 4$, correct to 2 decimal places?

A. 0.95

B. 1.26

C. 1.53

D. 2.84

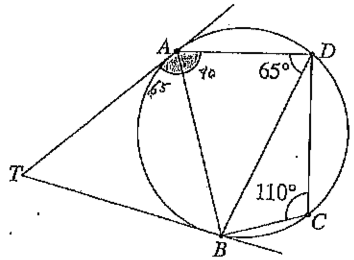
$3 \log 2 = 1.893$

$2 \log 2 = \frac{1.893}{3} \times 2$

$=$

3 The points A, B, C and D lie on a circle and the tangents at A and B meet at T, as shown in the diagram.

The angles BDA and BCD are $65°$ and $110°$ respectively.



What is the value of $\angle TAD$?

A. $130°$

B. $135°$

C. $155°$

D. $175°$

4 What is the value of $\tan\alpha$ when the expression $2\sin x - \cos x$ is written in the form $\sqrt{5}\sin(x-\alpha)$?

$\tan\alpha = \frac{1}{2}$

$\alpha = \tan^{-1}(\frac{1}{2})$

A. -2

B. $-\frac{1}{2}$

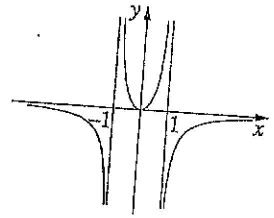
C. $\frac{1}{2}$

D. 2

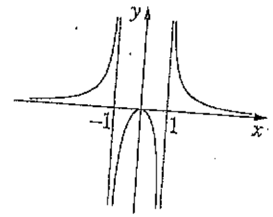
5 Which graph best represents the function $y = \dfrac{2x^2}{1-x^2}$?

$=$ $-x^2+1\,\overline{)\,2x^2}$ -2 $2x^2$

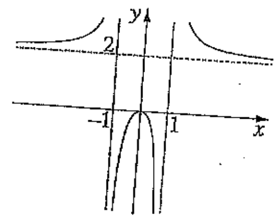
A.



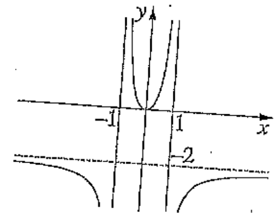
B.



C.



D.



6 The point $P\left(\dfrac{2}{p}, \dfrac{1}{p^2}\right)$, where $p \neq 0$, lies on the parabola $x^2 = 4y$.

What is the equation of the normal at P?

$y' = \frac{2x}{4} = \frac{\frac{2}{p}}{2} = \frac{1}{p}$

$y - \frac{1}{p^2} = -p\left(x - \frac{2}{p}\right)$

$p^2y - 1 = -p^3\left(x - \frac{2}{p}\right)$

$= -p^3x + 2p^2$

A. $py - x = -p$

B. $p^2y + px = -1$

C. $p^2y - p^3x = 1 - 2p^2$

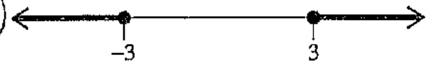
D. $p^2y + p^3x = 1 + 2p^2$

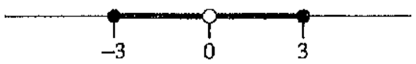
7 Which diagram represents the domain of the function $f(x) = \sin^{-1}\left(\frac{3}{x}\right)$? $-\frac{\pi}{2} \le f(x) \le \frac{\pi}{2}$

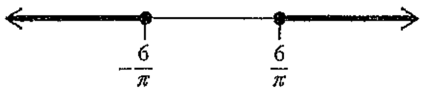
$\sin \frac{3}{4}$

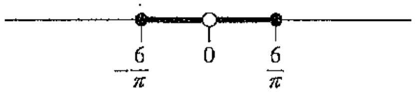
$\sin^{-1}(-1) = -\frac{\pi}{2}$

$\left(-\frac{3}{4}\right)$

A. 

B. 

C. 

D. 

8 A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of 5 cm s^{-1}.

$A = \pi r^2$

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. $25\pi \text{ cm}^2 \text{s}^{-1}$

B. $30\pi \text{ cm}^2 \text{s}^{-1}$

C. $150\pi \text{ cm}^2 \text{s}^{-1}$

D. $225\pi \text{ cm}^2 \text{s}^{-1}$

$\frac{dr}{dt} = 5$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

$= 2\pi r \cdot 5$

$= 10\pi(15)$

$= 150\pi$

9 When expanded, which expression has a non-zero constant term?

A. $\left(x + \frac{1}{x^2}\right)^7$

B. $\left(x^2 + \frac{1}{x^3}\right)^7$

C. $\left(x^3 + \frac{1}{x^4}\right)^7$

D. $\left(x^4 + \frac{1}{x^5}\right)^7$

$T_{r+1} = {}^7C_r \, a^{7-r} b^r$

$(x^4)^{7-r}(x^{-5})^r = 0$

$28 - 4r - 5r = 0$

$(x^3)^{7-r}(x^{-4})^r$

$21 - 3r - 4r = 0$

$7r = 21$

Three squares are chosen at random from the 3×3 grid below, and a cross is place[d in] each chosen square.

What is the probability that all three crosses lie in the same row, column or diagonal?

A. $\frac{1}{28}$

B. $\frac{2}{21}$

C. $\frac{1}{3}$

D. $\frac{8}{9}$

$\frac{8}{{}^9C_3} = \frac{8 \times 3 \times 2}{9 \times 8 \times 7}$

$= \frac{2}{21}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval from $A(-4,-4)$ to $B(1,6)$ internally in the ratio $2:3$. **1**

Find the x-coordinate of P.

$x = -2$

$x = \frac{-12+2}{5} = -\frac{10}{5} = -2$

(b) Differentiate $\tan^{-1}(x^3)$. **2**

$y' = 3x^2 \cdot \frac{1}{1+x^6} = \frac{3x^2}{1+x^6}$

(c) Solve $\frac{2x}{x+1} > 1$. **3**

$\Rightarrow \frac{2x-x-1}{x+1} > 0$

$\frac{x-1}{x+1} > 0$

$\boxed{x > 1 \text{ or } x < -1}$

(d) Sketch the graph of the function $y = 2\cos^{-1}x$. **2**

(e) Evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$. **3**

$x = 3,\ u = 2$
$x = 0,\ u = 1$
$\frac{dx}{du} = 2u$

$\int_1^2 \frac{u^2-1}{u} \cdot 2u\,du$

$= 2\left[\frac{u^3}{3} - u\right]_1^2 = 2\left[\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right]$

$= 2\left[\frac{7}{3} - 1\right]$

$= 2 \cdot \frac{4}{3} = \frac{8}{3}$

(f) Find $\int \sin^2 x \cos x\,dx$.

$u = \sin x$
$\frac{du}{dx} = \cos x$

$= \int u^2\,du$

$= \frac{u^3}{3} + c$

$= \frac{\sin^3 x}{3} + c$

Question 11 continues on page 8

(g) The probability that a particular type of seedling produces red flowers is $\frac{1}{5}$. Eight of these seedlings are planted.

(i) Write an expression for the probability that exactly three of the eight seedlings produce red flowers.

${}^8C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^5$

(ii) Write an expression for the probability that none of the eight seedlings produces red flowers.

$\left(\frac{4}{5}\right)^8$

(iii) Write an expression for the probability that at least one of the eight seedlings produces red flowers.

$1 - \left(\frac{4}{5}\right)^8$

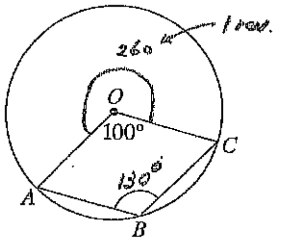
End of Question 11

$p = \frac{1}{5}$ $q = \frac{4}{5}$

$(p+q)^8$

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle AOC$ is $100°$. **2**



NOT TO SCALE

Find the size of $\angle ABC$, giving reasons. $= 130°$ (∠ at centre = 2 ∠ on the circumf)

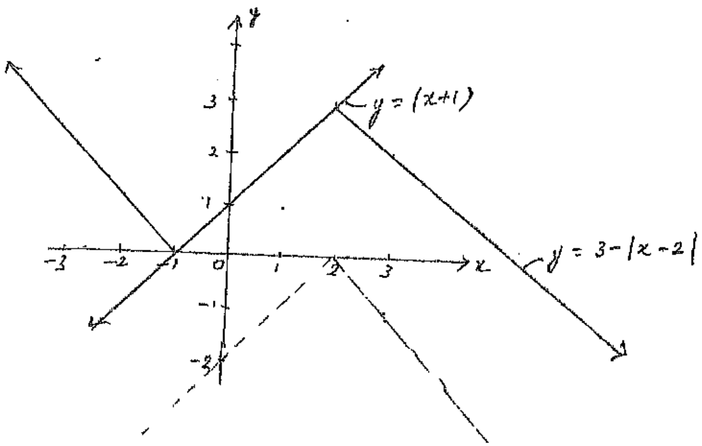
(b) (i) Carefully sketch the graphs of $y=|x+1|$ and $y=3-|x-2|$ on the same axes, showing all intercepts. **3**

(ii) Using the graphs from part (i), or otherwise, find the range of values of x for which **1**

$$|x+1|+|x-2|=3. \Rightarrow |x+1| = 3-|x-2|$$

$$\boxed{-1 \le x \le 2}$$

Question 12 continues on page 10



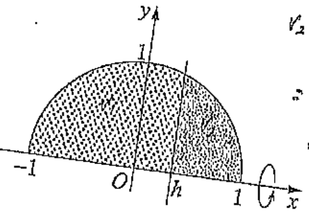
(c) The region enclosed by the semicircle $y=\sqrt{1-x^2}$ and the x-axis is to be divided into two pieces by the line $x=h$, where $0 \le h < 1$.

$V_1 = \pi \int_{-1}^{h} 1-x^2 \, dx$

$= \pi \left[x - \frac{x^3}{3}\right]_{-1}^{h}$

$= \pi \left[\left(h - \frac{h^3}{3}\right) - \left(-1 + \frac{1}{3}\right)\right]$

$= \pi \left[h + \frac{2}{3} - \frac{h^3}{3}\right]$



$V_2 = \pi \int_{h}^{1} 1-x^2 \, dx$

$= \pi \left[\left(1 - \frac{1}{3}\right) - \left(h - \frac{h^3}{3}\right)\right]$

$= \pi \left[\frac{2}{3} - h + \frac{h^3}{3}\right]$

The two pieces are rotated about the x-axis to form solids of revolution. The value of h is chosen so that the volumes of the solids are in the ratio $2:1$.

$\frac{V_1}{V_2} = \frac{2}{1}$

$\frac{3h+2-h^3}{2-3h+h^3} = \frac{2}{1}$

(i) Show that h satisfies the equation $3h^3-9h+2=0$.

$3h+2-h^3 = 4-6h+2h^3$

$3h^3 - 9h + 2 = 0$ as req'd

(ii) Given $h_1=0$ as the first approximation for h, use one application of Newton's method to find a second approximation for h.

$f(h) = 3h^3 - 9h + 2 \Rightarrow f(0) = 2$

$f'(h) = 9h^2 - 9 \Rightarrow f'(0) = -9$

$\therefore h_2 = h_1 - \frac{f(h_1)}{f'(h_1)}$

$= 0 - \frac{2}{-9}$

$= \frac{2}{9}$

(d) At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$.

Find the acceleration of the particle as a function of x. **3**

$e^{-2x} = 4-t$

$e^{2x} = \frac{1}{4-t}$

$2x = \ln\left(\frac{1}{4-t}\right)$

$x = \frac{1}{2}\ln\frac{1}{4-t}$

$\dot{x} = \frac{1}{2} \cdot \frac{-1}{4-t}$

$= \frac{-1}{8-2t} = -(8-2t)^{-1}$

$\ddot{x} = 1(8-2t)^{-2} \cdot (-2)$

$= \frac{-2}{(8-2t)^2}$

$= \frac{-2}{(8-2(4-e^{-2x}))^2}$

$= \frac{-2}{(2e^{-2x})^2}$

$= \frac{-2}{4e^{-4x}}$

$= \frac{-1}{2e^{-4x}}$

(e) Evaluate $\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$. **2**

$\cos 2x = 1 - 2\sin^2 x$

$\Downarrow$

$\lim_{x \to 0} \frac{2\sin^2 x}{x^2}$

$= 2$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the x-axis in simple harmonic motion centred at the origin. **3**

When $x = 2$ the velocity of the particle is 4.

When $x = 5$ the velocity of the particle is 3.

Find the period of the motion.

$v^2 = n^2(a^2 - x^2)$

$4^2 = n^2(a^2 - 4) \quad \cdots (1)$

$3^2 = n^2(a^2 - 25) \quad (2)$

$\frac{16}{9} = \frac{a^2-4}{a^2-25}$

$16(a^2-25) = 9(a^2-4)$

$7a^2 = -36 + 400 = 364$

$a^2 = 52$

$a = \sqrt{52}$

$\therefore\ 9 = n^2(52-25)$

$n^2 = \frac{1}{3}$ $\quad n = \frac{1}{\sqrt{3}}$

$\therefore T = \underline{2\sqrt{3}\pi}$

(b) Let n be a positive EVEN integer.

(i) Show that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$. **2**

(ii) Hence show that **1**

$$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right].$$

(iii) Hence show that $\binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n} = n2^{n-3}$. **2**

Question 13 continues on page 12

(i) $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 \ldots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

$+\ (1-x)^n = {}^nC_0(-x)^0 + {}^nC_1(-x)^1 + {}^nC_2(-x)^2 \ldots + {}^nC_{n-1}(-x)^{n-1} + {}^nC_n(-x)^n$

$= 2\,{}^nC_0 + 2\,{}^nC_2 x^2 + \ldots + 2\,{}^nC_n x^n$ when n is even

$= 2\left[{}^nC_0 + {}^nC_2 x^2 + \ldots + {}^nC_n x^n\right]$

(ii) Diff (i) w.r.t. x

$n(1+x)^{n-1} + n(1-x)^{n-1} \cdot -1 = 4\,{}^nC_2 x + 8\,{}^nC_4 x^3 + \ldots + 2n\,{}^nC_n x^{n-1}$

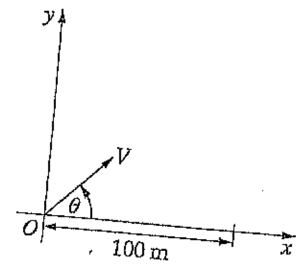
$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\,{}^nC_2 x + 4\,{}^nC_4 x^3 + \ldots + n\,{}^nC_n x^{n-1}\right]$

(iii) Let $x = 1$

$n \cdot 2^{n-1} = 2 \cdot 2\left[{}^nC_2 + 2\,{}^nC_4 + \ldots + \frac{n}{2}\,{}^nC_n\right]$

$n \cdot \frac{2^{n-1}}{4} = n \cdot 2^{n-3}$

(c) A golfer hits a golf ball with initial speed $V\ \mathrm{m\,s^{-1}}$ at an angle θ to the horizontal. The golf ball is hit from one side of a lake and must have a horizontal range of 100 m or more to avoid landing in the lake.



Neglecting the effects of air resistance, the equations describing the motion of the ball are

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time in seconds after the ball is hit and g is the acceleration due to gravity in $\mathrm{m\,s^{-2}}$. Do NOT prove these equations.

(i) Show that the horizontal range of the golf ball is $\frac{V^2\sin 2\theta}{g}$ metres. **2**

(ii) Show that if $V^2 < 100g$ then the horizontal range of the ball is less than 100 m. **1**

It is now given that $V^2 = 200g$ and that the horizontal range of the ball is 100 m or more.

(iii) Show that $\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$. **2**

(iv) Find the greatest height the ball can achieve. **2**

For greatest height

$\dot{y} = 0 = V\sin\theta - gt$

$t = \frac{V\sin\theta}{g}$

$\therefore y_{max} = \frac{V\sin\theta \cdot V\sin\theta}{g} - \frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2$

$= \frac{V^2\sin^2\theta}{2g}$ $\quad \frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$ $\quad \theta = \frac{\pi}{4} = \frac{3\pi}{12}$ $\quad y_{max} = \frac{V^2}{2g} \cdot \frac{1}{2}$

$\boxed{= \frac{V^2}{4g}}$

End of Question 13

(i) $y = 0 = t\left(V\sin\theta - \frac{1}{2}gt\right)$ $\quad \therefore t = 0$ or $t = \frac{2V\sin\theta}{g}$

$\therefore \frac{x}{R} = V\cos\theta\left(\frac{2V\sin\theta}{g}\right) = \frac{V^2\sin 2\theta}{g}$ m. since $2\sin\theta\cos\theta = \sin 2\theta$

(ii) $V^2 < 100g$ $\quad \frac{V^2\sin 2\theta}{g} < \frac{100\sin 2\theta \cdot g}{g} < 100$ since $-1 \le \sin 2\theta \le 1$

(iii) $V^2 = 200g$ $\quad R \ge 100$

$\therefore 200\sin 2\theta \ge 100$ $\qquad 0 \le \theta \le \frac{\pi}{2}$

$\sin 2\theta \ge \frac{1}{2}$ $\qquad 0 \le 2\theta \le \pi$

$\pi/6 \le 2\theta \le \frac{5\pi}{6}$ $\qquad \boxed{\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}}$

Question 14 (15 marks) Use a SEPARATE writing booklet.

$8^{2k+1}+6^{2k-1}=7M$

$8^{2k+3}+6^{2k+1}=7N$

(a) Prove by mathematical induction that $8^{2n+1}+6^{2n-1}$ is divisible by 7, for any integer $n \geq 1$. **3**

$LHS = 8^{2k+1}\cdot 8^2 + 6^{2k-1}\cdot 6^2$

$= 64(7M - 6^{2k-1}) + 36\cdot 6^{2k-1} = 64.7M$

$- 64m + 36m$

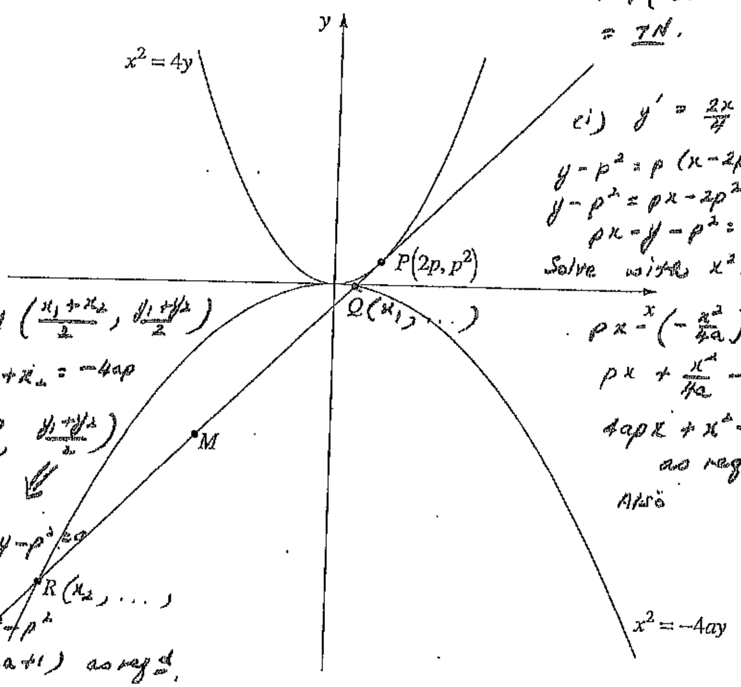
$- 28m$

$= 7(64M - 4m)$

$= 7N.$

(b) Let $P(2p, p^2)$ be a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P meets the parabola $x^2 = -4ay$, $a > 0$, at Q and R. Let M be the midpoint of QR.



(i) $y' = \frac{2x}{4} = \frac{4p}{4} = p$

$y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$px - y - p^2 = 0 \quad \cdots ①$

Solve with $x^2 = -4ay$

$px - \left(-\frac{x^2}{4a}\right) - p^2 = 0$

$px + \frac{x^2}{4a} - p^2 = 0$

$4apx + x^2 - 4ap^2 = 0$

as req'd.

Also

(ii) $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ $Q(x_1, \ldots)$

but $x_1 + x_2 = -4ap$

$\therefore \left(-\frac{4ap}{2}, \frac{y_1+y_2}{2}\right)$

Sub into the line

$p\left(-\frac{4ap}{2}\right) - y - p^2 = 0$ $R(x_2, \ldots)$

$\therefore y = -2ap^2 - p^2$

$= -p^2(2a+1)$ as req'd.

(i) Show that the x coordinates of R and Q satisfy **2**

$$x^2 + 4apx - 4ap^2 = 0.$$

(ii) Show that the coordinates of M are $(-2ap, -p^2(2a+1))$. **2**

(iii) Find the value of a so that the point M always lies on the parabola $x^2 = -4y$. **2**

(iii) $x = -2ap$, $y = -p^2(2a+1)$ $\Rightarrow$ $4a^2p^2 = +4p^2(2a+1)$

$4a^2 = 8a + 4$

$a^2 - 2a - 1 = 0$

$a = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$

Question 14 continues on page 14

$80e^{-1} \rightarrow 16.28$

$10 - 16.25$

$11.16 - 15.61$

Question 14 (continued)

(c) The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

	$t=2$	$t=2.23$	$t=3$
$F'(t)$	$=2.14$		-4.45

$\therefore$ Max.

(i) By differentiating the product $F(t)e^{0.4t}$ show that **2**

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. **2**

(iii) The concentration of the drug increases to a maximum. **2**

For what value of t does this maximum occur?

(iii) $F'(t) = 50e^{-0.5t} - 0.4 \times 500\left(e^{-0.4t} - e^{-0.5t}\right) = 0$

$= 50e^{-0.5t} - 200e^{-0.4t} + 200e^{-0.5t} = 0$

$250e^{-0.5t} = 200e^{-0.4t}$

$e^{-0.1t} = \frac{4}{5}$

$e^{0.1t} = \frac{5}{4} \Rightarrow 0.1t = \ln\left(\frac{5}{4}\right) \Rightarrow \boxed{t = 10\ln\left(\frac{5}{4}\right) \approx 2.23\text{ hrs}} \longrightarrow$ Test for max

End of paper

(i) $\frac{d}{dt}\left(F(t).e^{0.4t}\right) = F'(t).e^{0.4t} + 0.4F(t).e^{0.4t}$

$= \left[50e^{-0.5t} - 0.4F(t)\right]e^{0.4t}$

$+ 0.4F(t)e^{0.4t}$

$= 50e^{-0.1t} - 0.4F(t).e^{0.4t} + 0.4F(t)e^{0.4t}$

$= 50e^{-0.1t}$ as req'd.

(ii) $\int \frac{d}{dt}\left(F(t)e^{0.4t}\right)dt = \int 50e^{-0.1t}$

$F(t)e^{0.4t} = \frac{50e^{-0.1t}}{-0.1} + c$

At $t = 0$, $F(t) = 0$

$c = \frac{50}{0.1} = 500$

$\therefore F(t)e^{0.4t} = 500 - 500e^{-0.1t}$

$\therefore F(t) = 500\left(\frac{1 - e^{-0.1t}}{e^{0.4t}}\right)$

$= 500\left(e^{-0.4t} - e^{-0.5t}\right)$