SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

HSC Assessment Task 3

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes.
- Working time – 90 Minutes.
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each question in Section II is to be answered in a separate booklet.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest **EXACT** form unless otherwise instructed.
- A reference sheet is provided.

Total Marks – 61

Section I (7 Marks)

Answer questions 1-7 on the Multiple Choice answer sheet provided.

Section II (54 Marks)

For Questions 8-10, start a new answer booklet for each question.

Examiner: J. Chan

Section I

10 marks
Attempt Questions 1-7

Use the multiple-choice answer sheet for Questions 1-7

1) What is the domain of the function $f(x) = 5\sin^{-1}\left(\frac{x}{3}\right)$?

(A) $-\frac{5\pi}{2} \le x \le \frac{5\pi}{2}$

(B) $-3 \le x \le 3$

(C) $-5 \le x \le 5$

(D) $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

2) Which of the following is the exact value of $\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4}{\sqrt{9-x^2}}\,dx$?

(A) $-\pi$

(B) $-\frac{\pi}{4}$

(C) $\frac{\pi}{4}$

(D) π

3) Given $f(x) = \frac{3}{x} - 4$, $f^{-1}(4) = ?$

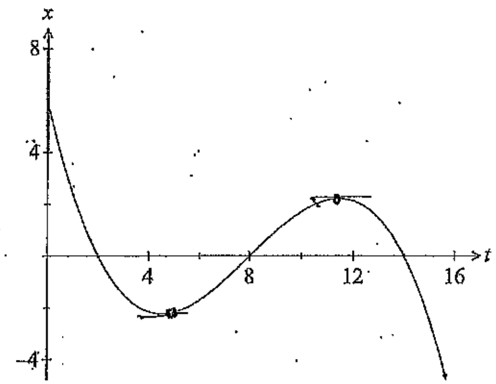
(A) $-\frac{13}{4}$

(B) $\frac{13}{4}$

(C) $\frac{3}{8}$

(D) $-\frac{3}{8}$

4) The displacement, x metres, from the origin of a particle moving in a straight line at any time (t seconds) is shown in the graph.



When was the particle at rest?

(A) $t=0$

(B) $t=2$, $t=8$ and $t=14$

(C) $t=5$ and $t=11$

(D) $t=8$

5) Using the substitution $u=\log_e x$, which of the following is equal to $\int_e^{e^2} \frac{1}{x\log_e x}\,dx$?

(A) $\int_e^{e^2} \frac{du}{u}$

(B) $\int_e^{e^2} \frac{du}{e^u u}$

(C) $\int_1^2 \frac{du}{u}$

(D) $\int_1^2 \frac{du}{e^u u}$

6) The acceleration of a particle is given by $a=6x^2-4x-3$, where x is the displacement in cm. The particle initially is at the origin and has a velocity of 3 cm/s. What is the speed when the particle is at $x=3$?

(A) $2\sqrt{7}$ cm/s

(B) $3\sqrt{7}$ cm/s

(C) $\sqrt{41}$ cm/s

(D) $\sqrt{57}$ cm/s

7) What is the value of $\cos^{-1}\left(\cos(3\pi+\alpha)\right)$ where α is an acute angle?

(A) $3\pi+\alpha$

(B) α

(C) $\pi+\alpha$

(D) $\pi-\alpha$

End of Section I

Section II

54 marks
Attempt Questions 8–10

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

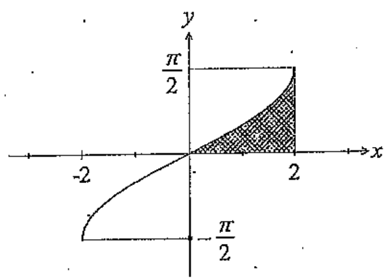
In Questions 8-10, your responses should include relevant mathematical reasoning and/or calculations.

Question 8 (16 marks) Use a SEPARATE writing booklet.

(a) Find the primitive of $2\sin^2 x - x^2$ **2**

(b) Show that $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) = \cos^{-1}x$ **2**

(c) The shaded area is represented by $\int_0^2 \sin^{-1}\frac{x}{2}\,dx$. **2**



Explain why the shaded area is equal to $\pi - 2\int_0^{\frac{\pi}{2}} \sin y\,dy$.

Question 8 continues next page

(d) Consider the function $f(x) = 1 + \frac{2}{x-3}$ for $x > 3$.

i) Find the inverse function $f^{-1}(x)$. **1**

ii) State the domain and range of the inverse function. **2**

iii) Hence sketch $y = f^{-1}(x)$. **1**

(e) The two equal sides of an isosceles triangle are of length 6 cm. If the angle between them is increasing at the rate of 0.05 radians per second, find the rate at which the area of the triangle is increasing when the angle between the equal sides is $\frac{\pi}{6}$ radians. **3**

(f) A particle moves in a straight line so that at any time t, its displacement from a fixed point is x and its velocity is v. **3**

If the acceleration is $3x^2$ and $v = -\sqrt{2}$, $x = 1$ when $t = 0$, find x as a function of t.

End of Question 8

Question 9 (19 marks) Use a SEPARATE writing booklet.

(a) i) Show that $\dfrac{u^3}{u+1}=u^2-u+1-\dfrac{1}{u+1}$ **1**

ii) Hence, by using the substitution $u=\sqrt{x}$, show that **3**

$$\int_0^4 \frac{x}{1+\sqrt{x}}\,dx=\frac{16}{3}-\ln 9$$

(b) i) Use the derivative of $\cos\theta$ to show that **1**

$$\frac{d}{d\theta}(\sec\theta)=\sec\theta\tan\theta$$

ii) Use the substitution $x=\sec\theta-1$ to find the exact value of **3**

$$\int_{\sqrt{2}-1}^{1}\frac{dx}{(x+1)\sqrt{x^2+2x}}$$

(c) A particle, moving along the x-axis, starts at the origin with an initial velocity v_0.

Its acceleration is given by $\dfrac{d^2x}{dt^2}=4x^3-16x$.

i) Show that the quantity $E=\dfrac{1}{2}\left(\dfrac{dx}{dt}\right)^2-x^4+8x^2$ does not change with time. **1**

ii) Given that initial velocity $v_0=\sqrt{\dfrac{31}{8}}$, find the value of E. **1**

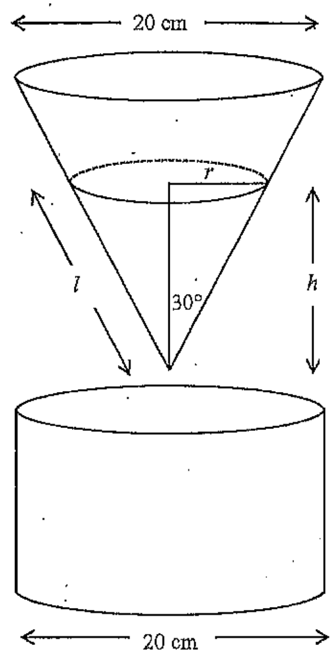
iii) Hence, or otherwise, determine the range of the particle with the initial velocity in part (ii). **2**

Question 9 continues next page

(d) An experiment is conducted using the conical filter which is held with its axis vertical as shown. The filter has a radius of 10 cm and semi-vertical angle 30°.

Chemical solution flows from the filter into the cylindrical container, with radius 10 cm, at a constant rate of 3 cm^3/s.

At time t seconds, the amount of solution in the filter has height h cm and radius r cm.



The volume of a cone of radius r and height h is $\dfrac{1}{3}\pi r^2 h$ and the curved surface area is $\pi r l$ where l is the slant height of the cone.

i) Find the rate of decrease (to 3 sig. fig.) of the radius of the solution in the filter when $h=5$ cm. **3**

ii) Let S denote the curved surface area of the filter in contact with the solution. **2**

Show that $\dfrac{dS}{dt}=-\dfrac{4\sqrt{3}}{r}$ cm^2/s.

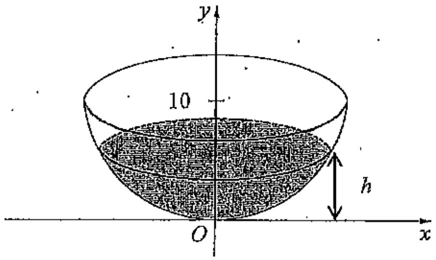
iii) When the height of the solution in the cylindrical container measures 0.81cm, the volume of the solution left in the filter and the container are the same. Find the rate of change with the curved surface area of the filter in contact with the solution with respect to time at this instant. **2**

End of Question 9

Question 10 (19 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\sin\left[\tan^{-1}\left(\frac{3}{2}\right)+\cos^{-1}\left(\frac{2}{3}\right)\right]$ **3**

(b) A hemispherical bowl of radius 10 cm is being filled with water at a constant rate of 20 cm^3 per minute.



NOT TO SCALE

i) Show that the volume of the water in the bowl in terms of its depth h is **3**

$$V=\pi\left(10h^2-\frac{h^3}{3}\right).$$

ii) At what rate (to 2 dec.pl.) is the depth of the water rising when it is 5 cm high? **3**

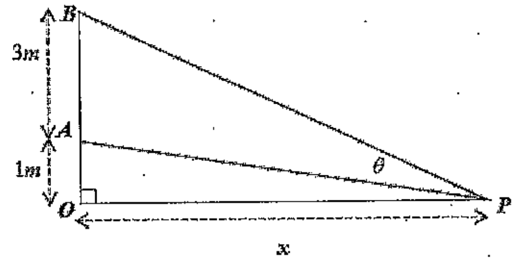
c) Let P_n denote the proposition $\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right)=\ln\frac{(n+1)^n}{n!}$. **4**

Prove by mathematical induction that this proposition is true for all positive integers, n.

Question 10 continues next page

Question 10 (continued)

(d) In the diagram, a vertical pole A, 3 metres high is placed on top of a support 1 metre high. The pole subtends an angle of θ radians at the point P, which is x metres from the base O of the support.



i) Show that $\theta=\tan^{-1}\frac{4}{x}-\tan^{-1}\frac{1}{x}$. **1**

ii) Show that θ is maximum when $x=2$. **3**

iii) Deduce that the maximum angle subtended at P is $\theta=\tan^{-1}\frac{3}{4}$. **2**

End of paper

SBHS Assessment task 3.

Ext 1

Multiple Choice.

1) $-1 \leq \frac{x}{3} \leq 1$

$-3 \leq x \leq 3.$

$\Rightarrow B$

2) $\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4}{\sqrt{9-x^2}}\, dx$

$= \left[4\left(\sin^{-1}\left(\frac{x}{3}\right)\right)\right]_{\frac{3}{\sqrt{2}}}^{3}$

$= 4\left[\sin^{-1}(1) - \sin^{-1}\left(\frac{3}{\sqrt{2}}/3\right)\right]$

$= \pi$

$\Rightarrow D$

3. $f(x) = \frac{3}{x} - 4$

$y = \frac{3}{x} - 4$

$y+4 = \frac{3}{x}$

$\frac{(y+4)}{3} = \frac{1}{x}.$

$\frac{3}{(y+4)} = x.$

$\therefore f^{-1}(x) = \frac{3}{(x+4)}$

$f^{-1}(4) = \frac{3}{4+4}$

$= \frac{3}{8}$

$\Rightarrow C.$

4. Particle at rest when $v=0$.

ie the turning points at $t=5$ and $t=11$

$\Rightarrow C.$

5. $\int_{e}^{e^2} \frac{1}{x \ln x}\, dx$

let $u = \ln x$.

$\frac{du}{dx} = \frac{1}{x}$

$x\, du = dx.$

$\int_1^2 \frac{1}{xu} \cdot x\, du.$

$= \int_1^2 \frac{1}{u}\, du.$

$= \int_1^2 \frac{du}{u}$

$\Rightarrow C.$

6) $a = 6x^2 - 4x - 3$

at $t=0$, $x=0$, $V=3$.

we want to find V at $x=3$.

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Integrate both sides.

$\int 6x^2 - 4x - 3\, dx = \int \left[\frac{d}{dx}\left(\frac{1}{2}v^2\right)\right] dx.$

$2x^3 - 2x^2 - 3x + C = \frac{v^2}{2}.$

$C = \frac{(3)^2}{2} = \frac{9}{2}.$

ie $\frac{v^2}{2} = 2x^3 - 2x^2 - 3x + \frac{9}{2}.$

$V^2 = 2\left(2(3)^3 - 2(3)^2 - 3(3) + \frac{9}{2}\right).$

$= 2\left(2(27) - 2(9) - 9 + \frac{9}{2}\right)$

$V^2 = 2(31.5) = 63$

$V = \sqrt{63}.$

$= \sqrt{7}\sqrt{9}$

$= 3\sqrt{7} \Rightarrow B.$

7). $\cos^{-1}(\cos(3\pi + \alpha))$

$\alpha < \frac{\pi}{2}.$

let $\alpha = 45° = \frac{\pi}{4}.$

$= 135°.$

$\underbrace{180°}_{\pi} - \underbrace{45°}_{\alpha} = 135°$

$= \pi - \alpha \Rightarrow D$

8. a) $\int 2\sin^2 x - x^2.$

$\int \frac{1}{2}(1 - \cos 2x)\, dx - \int x^2\, dx$

$= \frac{1}{2}\int (1 - \cos 2x)\, dx - \int x^2\, dx.$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] - \frac{x^3}{3} +$

$= \frac{x}{2} - \frac{1}{4}\sin(2x) - \frac{x^3}{3} + C.$

b) $\frac{d}{dx}\left(\underset{①}{x\cos^{-1}x} - \underset{②}{\sqrt{1-x^2}}\right)$

For pt ①, use product rule

$u = x$, $v = \cos^{-1}x$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}.$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

Pt 2 we use chain rule

$(1-x^2)^{\frac{1}{2}}$

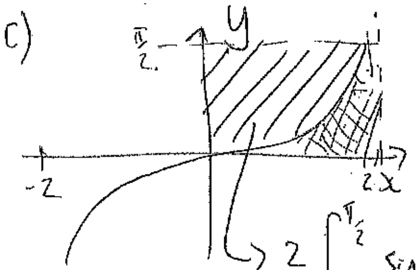
$= \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$

$= \frac{-x}{\sqrt{1-x^2}}$

combing parts (1) and (2)

$\cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \cos^{-1}(x)$

c)



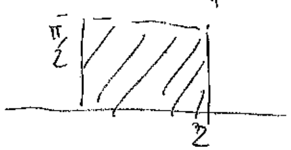
$\to 2\int_0^{\frac{\pi}{2}} \sin(y)\,dy$

In the 'x' equation

$y = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin(y) = \frac{x}{2}$, $x = \boxed{2\sin(y)}$

and the area of the rectangle



is $\frac{\pi}{2} \times 2 = \pi u^2$.

So we have the rectangle minus the y 'bit'

$= \pi - 2\int_0^{\frac{\pi}{2}} \sin y\,dy$.

d) $f(x) = 1 + \frac{2}{x-3}$ $x > 3$.

$y = 1 + \frac{2}{x-3}$.

$y - 1 = \frac{2}{x-3}$.

$\frac{y-1}{2} = \frac{1}{x-3}$.

$\frac{2}{y-1} = x - 3$.

$\frac{2}{y-1} + 3 = x$.

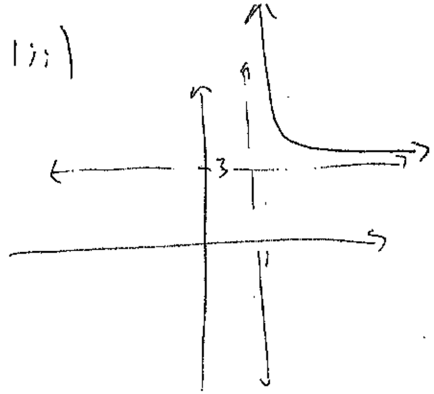
$f^{-1}(x) = \frac{2}{x-1} + 3$.

ii) Domain: $x \in \mathbb{R}$; $x \neq 1$

If we are taking the portion from pt i) and reflecting through $x=y$ axis, then domain is $x > 1$

Range: $y \in \mathbb{R}$; $y \neq 3$. If we do it to the same portion in pt i) then $y > 3$.

iii)



e)

Area $A = \frac{1}{2}ab\sin\theta$.

$= \frac{1}{2}a^2\sin C$, since lengths of isoceles Δ are equal.

$\frac{d\theta}{dt} = 0.05$

$\frac{d\theta}{dt} \times \frac{dA}{d\theta} = \frac{dA}{dt}$.

$A = 18\sin\theta$

$\frac{dA}{d\theta} = 18\cos\theta$

$0.05 \times 18\cos\theta = 0.9\cos\theta$

at $\theta = \frac{\pi}{6}$.

$dA = 0.9\cos\frac{\pi}{6}$

$\approx 0.78\text{cm}^2/\text{s}$.

f) $a = 3x^2 = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Integrate both sides.

$\int 3x^2\,dx = \frac{1}{2}v^2$.

$x^3 + C = \frac{1}{2}v^2$.

at $x = 1$, $v = \sqrt{2}$.

$1 + C = \frac{1}{2}(-\sqrt{2})^2$.

$C = 0$.

$x^3 = \frac{1}{2}v^2$.

$2x^3 = v^2$

$v = \pm\sqrt{2x^3}$

$\frac{dx}{dt} = \pm\sqrt{2x^3}$.

$dx = \pm\sqrt{2x^3}\,dt$.

$\frac{dx}{\pm\sqrt{2x^3}} = dt$.

$\int \frac{1}{\sqrt{2x^3}}\,dx = t$.

FACTOR OUT AND REVERSE CHAIN RULE

$= \frac{x\sqrt{2}}{\sqrt{x^3}} + c = t.$

at $t=0$, $x=1$

$\frac{\sqrt{2}}{1} + c = 0.$

$c = -\sqrt{2}.$

$\frac{x\sqrt{2}}{\sqrt{x^3}} - \sqrt{2} = t$

9. a) $\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$

From RHS.

$\frac{(u^2-u+1)(u+1)}{(u+1)} - \frac{1}{u+1}$

$= \frac{u^3+u^2-u^2-u+u+1}{(u+1)} - \frac{1}{(u+1)}$

$= \frac{u^3}{u+1} = \text{LHS}.$

ii) let $u = \sqrt{x} = (x)^{\frac{1}{2}}$

$\int_0^4 \frac{x}{1+\sqrt{x}}.$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ $\quad du(2\sqrt{x}) = dx$

$\int_0^2 \frac{u^2 \, 2\sqrt{x}}{1+u} = 2\int_0^2 \frac{u^3}{1+u}.$

by the substitution shown in i)

$= 2\int_0^2 u^2 - u + 1 - \frac{1}{u+1}$

$= 2\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1)\right]_0^2$

$2\left[\frac{8}{3} - 2 + 2 - \ln 3\right]$

$= 2\left(\frac{8}{3} - \ln 3\right)$

$= \frac{16}{3} - \ln 3^2$

$= \frac{16}{3} - \ln 9.$

b) $\frac{d}{d\theta}\cos\theta = -\sin\theta$

RTP $\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta.$

$= \frac{d}{d\theta}\frac{1}{\cos\theta} = \frac{d}{d\theta}(\cos\theta)^{-1}$

$= -1(\cos\theta)^{-2}(-\sin\theta)$

$= \frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}$

$= \tan\theta\sec\theta$

ii) let $x = \sec\theta - 1$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta \, d\theta}{(\sec\theta)\sqrt{\sec^2\theta - 1}}$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta \, d\theta}{\sec\theta\sqrt{\tan^2\theta}}$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 \, d\theta$

$= [\theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

$1 = \sec\theta - 1$
$\sec\theta = 2.$
$\frac{1}{\cos\theta} = 2.$
$\cos\theta = \frac{1}{2}$
$\theta = \cos^{-1}\left(\frac{1}{2}\right)$
$= \frac{\pi}{3}$

$\sqrt{2} - 1 = \sec\theta - 1.$
$\sqrt{2} = \sec\theta.$
$\theta = \cos^{-1}\frac{1}{\sqrt{2}}$
$= \frac{\pi}{4}$

$\frac{dx}{d\theta} = \sec\theta\tan\theta.$
$dx = \sec\theta\tan\theta \, d\theta.$

$1 + \tan^2\theta = \sec^2\theta$
$\sec^2\theta - 1 = \tan^2\theta$

c) i)

$a = 4x^3 - 16x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x^3 - 16x$

$\frac{1}{2}v^2 = \int 4x^3 - 16x \, dx$

$\frac{1}{2}v^2 = x^4 - 8x^2 + c$

$E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - x^4 + 8x^2$

$= x^4 - 8x^2 + c - x^4 + 8x^2$

$= c$

$\therefore$ does not change with time.

ii) $\frac{1}{2}v^2 = x^4 - 8x^2 + c$ $\qquad v = \sqrt{\frac{31}{8}}$ when $t=0$.

and $x = 0$.

$\frac{1}{2} \times \frac{31}{8} = 0 + 0 + c$

$c = \frac{31}{16}$

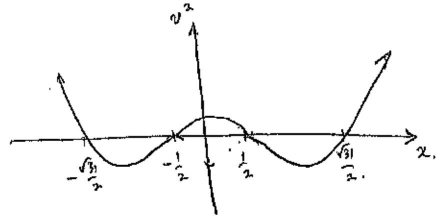
$\Rightarrow E = \frac{31}{16}$.

iii) $\frac{1}{2}v^2 = x^4 - 8x^2 + \frac{31}{16}$

$v^2 = 2x^4 - 16x^2 + \frac{31}{8}$

$8v^2 = 16x^4 - 128x^2 + 31$

$= (4x^2 - 31)(4x^2 - 1) > 0$



Since $x = 0$ when $t = 0$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$

9. d) i) $V = \frac{1}{3}\pi r^2 h$ $\qquad h = \frac{r}{\tan 30^\circ} = \sqrt{3} r$

$= \frac{1}{\sqrt{3}}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$\frac{dr}{dt} = \frac{-3}{\sqrt{3}\pi r^2}$

$\frac{dr}{dt} = -\frac{\sqrt{3}}{\pi(\frac{5}{2})^2}$ $\qquad$ When $h = 5$ $\quad r = \frac{5}{2}$

$= -0.0882$ cm/s

Decrease at 0.0882 cm/s

ii) $S = \pi r l$

$= \frac{\pi r^2}{\sin 30^\circ}$

$= 2\pi r^2$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= 4\pi r \times \frac{-\sqrt{3}}{\pi r^2}$

$= -\frac{4\sqrt{3}}{r}$ cm²/s

iii) $\frac{dS}{dt} = -\frac{4\sqrt{3}}{r}$ cm²/s $\qquad r = \frac{0.81}{\sqrt{3}}$

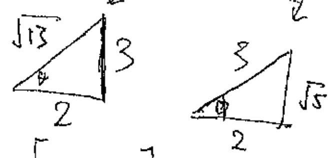
$= -\frac{12}{0.81}$

$= -14.8$ cm²/s

10. a)

$\sin\left[\tan^{-1}\left(\frac{3}{2}\right) + \cos^{-1}\left(\frac{2}{3}\right)\right]$ let $\tan^{-1}\left(\frac{3}{2}\right)$ be θ

and $\cos^{-1}\left(\frac{2}{3}\right)$ be ϕ



$\sin[\theta + \phi] = \sin\theta\cos\phi + \cos\theta\sin\phi$

$= \sin\left(\sin^{-1}\frac{3}{\sqrt{13}}\right)\cos\left(\cos^{-1}\frac{2}{3}\right) + \cos\left(\cos^{-1}\left(\frac{2}{\sqrt{13}}\right)\right)\sin\left(\sin^{-1}\frac{\sqrt{5}}{3}\right)$

$= \frac{3}{\sqrt{13}} \times \frac{2}{3} + \frac{2}{\sqrt{13}} \times \frac{\sqrt{5}}{3} = \frac{6}{3\sqrt{13}} + \frac{2\sqrt{5}}{3\sqrt{13}}$

$= \frac{6 + 2\sqrt{5}}{3\sqrt{13}}$ $\Rightarrow$ Rationalising. becomes

$\left[\frac{6\sqrt{13} + 2\sqrt{65}}{39}\right]$

6) i) we can use volume of solid of revolution

ie we have a semicircle rotated about Y-axis.

ie. the equation of the semicircle is.

$y = -\sqrt{10^2 - x^2} + 10$ (because shifted up 10 units).

$y - 10 = -\sqrt{10^2 - x^2} \quad y \geq 0$

$(y-10)^2 = 10^2 - x^2 \quad y \geq 0$

$(y^2 + 100 - 20y) = 10^2 - x^2$

$y^2 + \cancel{100} - 20y = \cancel{100} - x^2$

$x^2 = 20y - y^2$

∴ the resulting volume integral is therefore

$$\pi \int_0^h 20y - y^2 \, dy = \pi \left[\frac{20y^2}{2} - \frac{y^3}{3} \right]_0^h$$

$$= \pi \left[10y^2 - \frac{y^3}{3} \right]_0^h = \pi \left[10h^2 - \frac{h^3}{3} \right]$$

ii) $\frac{dV}{dh} = \pi\left(20h - \frac{h^2}{2}\right) = \pi\left(100 - \frac{25}{2}\right) = \frac{175}{2}\pi.$

$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt}$

$= \frac{2}{175\pi} \times 20$

$= 0.07$ cm/min

c) $P_n: \sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$

prove true for $n=1$.

$\sum_{r=1}^{1} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(1+1)^1}{1!}\right)$

$= 1\left(\ln\left(\frac{2}{1}\right)\right) = \ln\left(\frac{2}{1}\right)$

$\therefore$ true for $n=1$.

assume $n=k$, where $1 \leq k \leq n$.

$\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(k+1)^k}{k!}\right)$

Prove true for $n=k+1$ case

$\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(k+1+1)^{k+1}}{(k+1)!}\right)$

ie we want to prove.

$$\underbrace{\ln\left(\frac{(k+1+1)^{k+1}}{(k+1)!}\right) - \ln\left(\frac{(k+1)^k}{k!}\right)}_{} = (k+1)\ln\left(\frac{k+2}{k+1}\right)$$

we can use log laws

$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$$\text{LHS} = \ln\left(\frac{(k+2)^{k+1}}{(k+1)!} \times \frac{k!}{(k+1)^k}\right) = \ln\left(\frac{(k+2)^k(k+2)\,k!}{(k+1)k!\,(k+1)^k}\right) = \ln\left(\frac{(k+2)^k(k+2)}{(k+1)^k(k+1)}\right)$$

$$= \ln\left(\frac{(k+2)^k}{(k+1)^k}\right) + \ln\left(\frac{k+2}{k+1}\right) = k\ln\left(\frac{k+2}{k+1}\right) + \ln\left(\frac{k+2}{k+1}\right) = (k+1)\ln\left(\frac{k+2}{k+1}\right)$$

$= \text{RHS}$

$\therefore$ by the principle of mathematical induction, the proposition P_n holds for all integers $n \geq r$

d) i) let $\angle APO$ be α.

$\alpha + \theta = \tan^{-1}\frac{4}{x}$.

$\alpha = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \theta = \tan^{-1}\frac{4}{x} - \alpha$.

$= \tan^{-1}\frac{4}{x} - \tan^{-1}\left(\frac{1}{x}\right)$.

ii) $\frac{d\theta}{dx}$ using chain rule. $\Rightarrow \frac{d}{dx}\tan^{-1}(4x^{-1})$

let $u = 4x^{-1}$ etc.

Should get

$\frac{-4}{x^2+4^2} - \frac{-1}{x^2+1} = \frac{d\theta}{dx}$.

maximum at $\frac{d\theta}{dx} = 0$

$\frac{-4}{x^2+4^2} + \frac{1}{x^2+1} = 0$

$\frac{1}{x^2+1} = \frac{4}{x^2+4^2}$

$x^2+16 = 4x^2+4$

$3x^2 = 12$

$x^2 = \frac{12}{3} = 4$

$x = 2, -2$.

To prove maximum,

we need to

$\frac{d^2\theta}{dx^2} \Rightarrow \frac{8x}{(x^2+16)^2} - \frac{2x}{(x^2+1)^2}$.

when $x=2$, $\frac{d^2\theta}{dx^2} < 0$

$\therefore$ must be a max

$\frac{d\theta}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}\right)$

$= \frac{1}{1+\frac{16}{x^2}} \times \frac{-4}{x^2}$

$- \frac{1}{1+\frac{1}{x^2}} \times \frac{-1}{x^2}$

$= \frac{-4}{x^2+16} + \frac{1}{1+x^2}$

d) iii) Max θ when $x = 2$

$$\theta = \tan^{-1} 2 - \tan^{-1} \frac{1}{2}$$

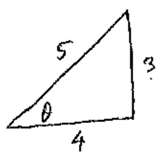
when $x = 2$,

$$PA = \sqrt{1+2^2} = \sqrt{5}$$

$$\frac{\sin\theta}{3} = \frac{\sin\angle ABP}{\sqrt{5}}$$

$$\sin\angle ABP = \frac{2}{\sqrt{2^2+4^2}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\theta = \sin^{-1} \frac{3}{5}$$



$$\theta = \tan^{-1} \frac{3}{4}.$$