2016 SYDNEY BOYS HIGH SCHOOL
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 3–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 8–15

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Examiner: E.C.

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Which integral is obtained when the substitution $u = 1 + 3x$ is applied to $\int x\sqrt{1+3x}\,dx$?

(A) $\frac{1}{9}\int (u-1)\sqrt{u}\,du$

(B) $\frac{1}{6}\int (u-1)\sqrt{u}\,du$

(C) $\frac{1}{3}\int (u-1)\sqrt{u}\,du$

(D) $\frac{1}{4}\int (u-1)\sqrt{u}\,du$

2 The acceleration of a particle moving along a straight line is given by $\ddot{x} = -2e^{-x}$, where x metres is the displacement from the origin.
If the velocity of the particle is v m/s, which of the following is a correct statement about v^2?

(A) $v^2 = 2e^{-x} + C$

(B) $v^2 = 2e^{x} + C$

(C) $v^2 = 4e^{-x} + C$

(D) $v^2 = 4e^{x} + C$

3 Find $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$

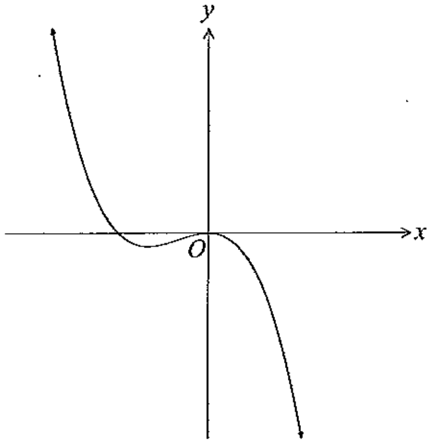
(A) $\frac{-2}{\sqrt{1-x^2}}$

(B) $\frac{-1}{\sqrt{1-x^2}}$

(C) $\cos^{-1}x$

(D) $\sin^{-1}x$

4 What is a possible equation of this function?



(A) $f(x)=-x(x-1)(x+1)$

(B) $f(x)=-x^2(x+1)$

(C) $f(x)=-x^2(x-1)$

(D) $f(x)=x^2(x+1)$

5 If $f(x)=1+\frac{2}{x-3}$, which of the following give the equations of the horizontal and vertical asymptotes of $f^{-1}(x)$?

(A) Vertical asymptote is $x=1$ and horizontal asymptote is $y=2$

(B) Vertical asymptote is $x=1$ and horizontal asymptote is $y=3$

(C) Vertical asymptote is $x=3$ and horizontal asymptote is $y=1$

(D) Vertical asymptote is $x=3$ and horizontal asymptote is $y=2$

6 The polynomial equation $x^3-ax^2+8x+(1-a)=0$ has roots α, β and γ.

Given that $\alpha+\beta+\gamma<0$ and $\alpha\beta\gamma(\alpha+\beta+\gamma)=20$, what is the value of a?

(A) –4

(B) 4

(C) –5

(D) 5

7 If $t=\tan\frac{\theta}{2}$, which of the following expressions is equivalent to $4\sin\theta+3\cos\theta+5$?

(A) $\frac{2(t+2)^2}{1-t^2}$

(B) $\frac{(t+4)^2}{1-t^2}$

(C) $\frac{2(t+2)^2}{1+t^2}$

(D) $\frac{(t+4)^2}{1+t^2}$

8 Which of the following is a correct expression for $\tan\left(x+\frac{\pi}{4}\right)$?

(A) $\frac{\cos x+\sin x}{\cos x-\sin x}$

(B) $\frac{\cos x+2\sin x}{\cos x-\sin x}$

(C) $\frac{\cos x+\sin x}{\cos^2 x-\sin x}$

(D) $\frac{\cos x-\sin x}{\cos x-\sin x}$

9 The curve $y = 2x^{\frac{1}{3}}$ is reflected in the line $y = x$.
What is the equation of the reflected curve?

(A) $y = \dfrac{x^3}{16}$

(B) $y = \dfrac{x^3}{8}$

(C) $y = \dfrac{x^3}{4}$

(D) $y = \dfrac{x^3}{2}$

10 A particle is moving in simple harmonic motion with displacement x.
Its velocity is given by $v^2 = 9(36 - x^2)$.
What is the amplitude, A, of the motion and the maximum speed of the particle?

(A) $A = 3$ and maximum speed $v = 6$

(B) $A = 3$ and maximum speed $v = 18$

(C) $A = 6$ and maximum speed $v = 18$

(D) $A = 6$ and maximum speed $v = 6$

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

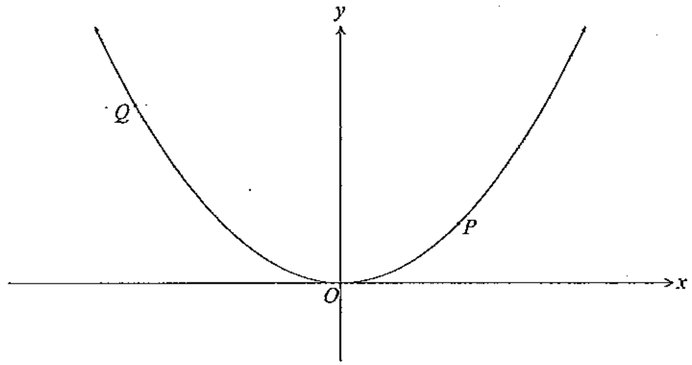
Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x: **2**

$$y = \tan^{-1}\left(\frac{2}{x}\right)$$

(b) Evaluate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-4x^2}}$ **2**

(c) The points $P\,(2ap, ap^2)$ and $Q\,(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ **2**



If the tangents at P and Q intersect at 45°, show that $|1 + pq| = |p - q|$.

Question 11 continues on page 9

Question 11 (continued)

(d) State the domain and range of the function $y=2\cos^{-1}3x$. **1**

(e) The roots of the equation $x^3-3x^2+4x+2=0$ are α, β, and γ. **2**
Find the value of $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$.

(f) Solve the equation $4\sin\theta+3\cos\theta=-5$ for $0°<\theta<360°$. **2**
Leave your answers correct to the nearest degree.

(g) (i) Show that the turning points of the curve $y=\dfrac{x}{(x+3)(x+4)}$ occur when $x=\pm 2\sqrt{3}$. **2**

(ii) Sketch $y=\dfrac{x}{(x+3)(x+4)}$ for $x\geq 0$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The rate at which perfume evaporates is proportional to the amount of the perfume that has not yet evaporated. That is $\dfrac{dN}{dt}=k(P-N)$, where P is the initial amount of perfume, N is the amount that has evaporated at time t and k is constant.

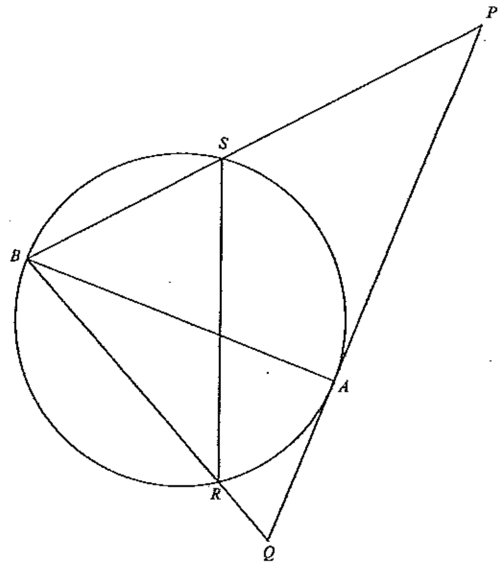
(i) Show that the function $N=P(1-e^{-kt})$ satisfies the differential equation **1**

$$\frac{dN}{dt}=k(P-N)$$

(ii) Show that the time it takes for a quarter of the original amount to evaporate is **2**

$$-\frac{(\ln 3-2\ln 2)}{k}$$

(b) In the diagram below, PAQ is the tangent to a circle at A.
AB is a diameter and lines PB and QB cut the circle at S and R respectively.



(i) Copy the diagram to your writing booklet.

(ii) Prove that $PQRS$ is a cyclic quadrilateral. **3**

Question 12 continues on page 11

Question 12 (continued)

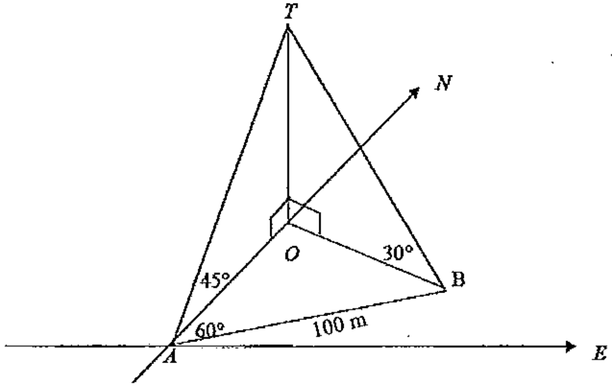
(c) In how many ways would 11 people occupy seats at two circular tables, where one table can accommodate 6 people and the other 5 people? **2**

(d) Consider the function $f(x)=3x-x^3$

(i) Find the largest domain containing the origin for which $f(x)$ has an inverse function $f^{-1}(x)$. **2**

(ii) State the domain of $f^{-1}(x)$. **1**

(e) To an observer on a pier A, the angle of elevation of the top of a cliff OT due North of the observer is 45°. After the observer travelled 100m by boat from the pier at N60°E to B, the angle of elevation of the top of the cliff is 30°.



Find the height of the cliff above the sea level. **2**

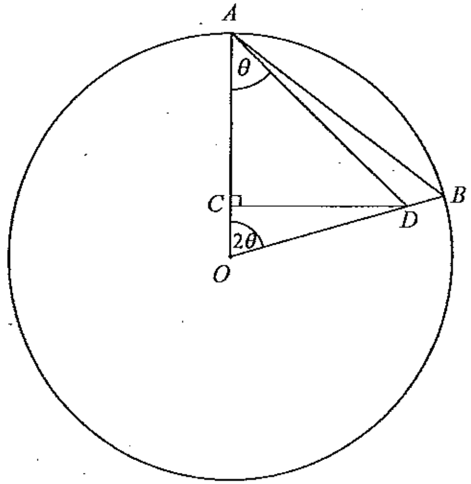
(f) A particle moves in a straight line with acceleration at any time t given by $\ddot{x}=-e^{-2x}$, where x metres is the distance measured from a fixed point O. **2**

Initially the particle is at the origin with velocity 1 m/s. Show that $x=\ln(t+1)$.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a circle of centre O and radius 1 m and $\angle AOD=2\theta$. D is a point on OB such that $\angle DAO=\theta$. Also, C is a point on OA such that $CD\perp OA$.



Let $CD=x$.

(i) Express AC in terms of x and θ, and by considering ΔOCD, show that **2**

$$x=\frac{2\tan\theta}{3-\tan^2\theta}.$$

(ii) If $x=\frac{\sqrt{3}}{4}$, find the value of θ, and hence, show that the area of $\Delta OAB=\frac{\sqrt{3}}{4}$ m². **2**

(b) Many calculators compute reciprocals by using the approximation $\frac{1}{a}\doteqdot x_{n+1}$, where $x_{n+1}=x_n(2-ax_n)$ for $n=1, 2, 3, \ldots$ **2**

That is if x_1 is an initial approximation to $\frac{1}{a}$, then $x_2=x_1(2-ax_1)$ is a better approximation.

This formula makes it possible to use multiplications and subtractions, which can be done quickly, to perform divisions that would be slow to obtain directly.

Apply Newton's method to $f(x)=\frac{1}{x}-a$, using x_1 as an initial approximation, to show

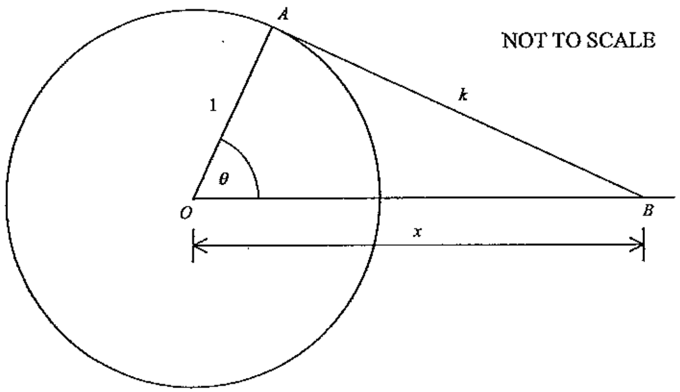
$$x_2=x_1(2-ax_1)$$

Question 13 continues on page 13

Question 13 (continued)

(c) Evaluate $\lim_{x\to 0} \frac{\sin 2x}{\tan 3x}$ **1**

(d) The diagram below shows a circular disc with radius OA.



The radius of the disc, OA, is one metre and AB is a rod of length k metres ($k > 1$).
The end of the rod, B, is free to slide along a horizontal axis with origin O.
The angle between OA and OB is θ.

Let $OB = x$ metres.

(i) Show that $x = \cos\theta + \sqrt{k^2 - \sin^2\theta}$. **3**

(ii) Find $\frac{dx}{d\theta}$ in terms of k and θ. **2**

(iii) Given that $\frac{d\theta}{dt} = 4\pi$ rad/s. **2**

Find $\frac{dx}{dt}$ in terms of k when $\theta = \frac{\pi}{6}$.

(iv) Find θ, $0 \le \theta < 2\pi$, when the velocity of point B is zero. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that $\sum_{r=1}^{n}(r+3)2^r = (n+2)2^{n+1} - 4$ **3**

where n is a positive integer.

(b) A particle performs simple harmonic motion on a straight line.
It has zero speed at the points A and B whose distances on the same side from a fixed point O are a and b respectively, where $b > a$.

(i) Find the amplitude of oscillation in terms of a and b. **1**

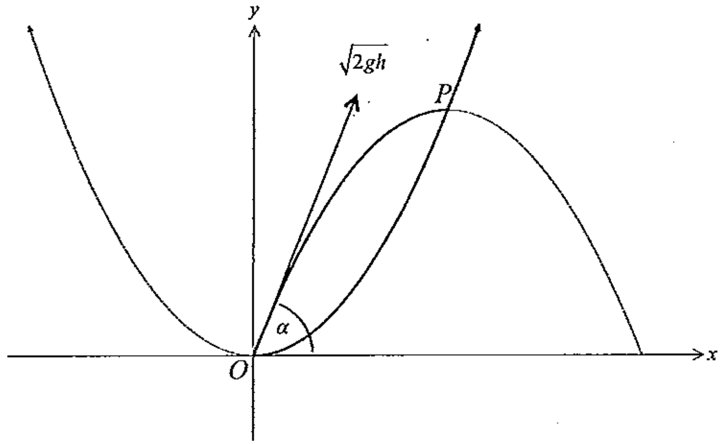
(ii) The particle has a speed V when half way between the points A and B. **3**
Show that the period of oscillation is $\frac{\pi(b-a)}{V}$.

You may use the following formula: $v^2 = n^2(c^2 - (x - x_0)^2)$
(Do **NOT** prove this)

Question 14 continues on page 15

Question 14 (continued)

(c) A vertical section of a valley is in the form of the parabola $x^2 = 4ay$ where a is a positive constant.
A gun placed at the origin fires with speed $\sqrt{2gh}$ at an angle of elevation α where $0 < \alpha < \frac{\pi}{2}$ and h is a positive constant.



The equations of the motion of a projectile fired from the origin with initial velocity $V\,\text{ms}^{-1}$ at angle θ to the horizontal are

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ (Do **NOT** prove this)

(i) If the shell strikes the section of the valley at the point $P(x, y)$ show that **3**

$$x = \frac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$$

(ii) Let $f(\theta) = (a+h)\cot\theta + a\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. **2**

Show that the minimum value of $f(\theta)$ occurs when $\tan\theta = \sqrt{\frac{a+h}{a}}$.

(iii) Show that the greatest value of x is given by **3**

$$x = 2h\sqrt{\frac{a}{a+h}}$$

End of paper

Section I

10 marks
Attempt Questions 1-10

Use the multiple-choice answer sheet for Questions 1-10

1 Which integral is obtained when the substitution $u = 1 + 3x$ is applied to $\int x\sqrt{1+3x}\,dx$?

(A) $\frac{1}{9}\int (u-1)\sqrt{u}\,du$

(B) $\frac{1}{6}\int (u-1)\sqrt{u}\,du$

(C) $\frac{1}{3}\int (u-1)\sqrt{u}\,du$

(D) $\frac{1}{4}\int (u-1)\sqrt{u}\,du$

$u = 1+3x$
$\frac{du}{dx} = 3$
$du = 3\,dx$
$\frac{du}{3} = dx$
$u = 1+3x$
$\frac{u-1}{3} = x$

$\int \frac{(u-1)}{3}\sqrt{u}\,.\frac{du}{3}$

$\frac{1}{9}\int (u-1)\sqrt{u}\,du$

2 The acceleration of a particle moving along a straight line is given by $\ddot{x} = -2e^{-x}$, where x metres is the displacement from the origin.
If the velocity of the particle is v m/s, which of the following is a correct statement about v^2?

(A) $v^2 = 2e^{-x} + C$

(B) $v^2 = 2e^{x} + C$

(C) $v^2 = 4e^{-x} + C$

(D) $v^2 = 4e^{x} + C$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

$\int \frac{d}{dx}\left(\frac{1}{2}v^2\right)dx = \int -2e^{-x}\,dx$

$\frac{1}{2}v^2 = 2e^{-x} + C$

$v^2 = 4e^{-x} + C.$

$(1-x^2)^{\frac{1}{2}}$

3 Find $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$

(A) $\frac{-2}{\sqrt{1-x^2}}$

(B) $\frac{-1}{\sqrt{1-x^2}}$

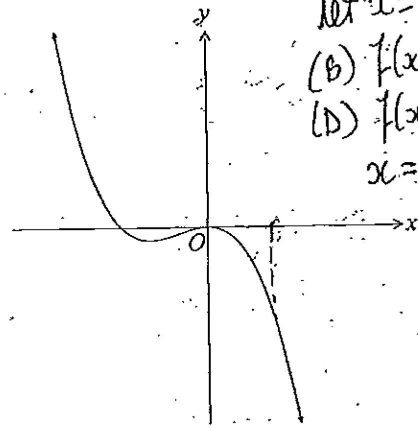
(C) $\cos^{-1}x$

(D) $\sin^{-1}x$

$x \times \frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$

$\cancel{\frac{-x}{\sqrt{1-x^2}}} + \cos^{-1}x + \cancel{\frac{x}{\sqrt{1-x^2}}}$

4 What is a possible equation of this function?



let $x = 2$ (say)
(B) $f(x) = -4\times3 = -12$
(D) $f(x) = 4\times3 = 12$ No!
$x = 0$ double root.
x negative single root.

(A) $f(x) = -x(x-1)(x+1)$

(B) $f(x) = -x^2(x+1)$

(C) $f(x) = -x^2(x-1)$

(D) $f(x) = x^2(x+1)$

5 If $f(x) = 1 + \frac{2}{x-3}$, which of the following give the equations of the horizontal and vertical asymptotes of $f^{-1}(x)$?

(A) Vertical asymptote is $x = 1$ and horizontal asymptote is $y = 2$

(B) Vertical asymptote is $x = 1$ and horizontal asymptote is $y = 3$

(C) Vertical asymptote is $x = 3$ and horizontal asymptote is $y = 1$

(D) Vertical asymptote is $x = 3$ and horizontal asymptote is $y = 2$

The vertical asymptote of $f(x)$ is $x = 3$
So its inverse, the horizontal asymptote is $y = 3$. Hence B.

6 The polynomial equation $x^3 - ax^2 + 8x + (1-a) = 0$ has roots α, β and γ.

Given that $\alpha+\beta+\gamma<0$ and $\alpha\beta\gamma(\alpha+\beta+\gamma)=20$, what is the value of a?

(A) -4
(B) 4
(C) -5
(D) 5

$a=1$
$b=-a$
$c=8$
$d=1-a$

$* \ \alpha+\beta+\gamma = -\frac{b}{a} = a$
$\alpha\beta+\alpha\gamma+\beta\gamma = 8$
$\alpha\beta\gamma = -\frac{d}{a} = a-1$

So
$(a-1)a = 20$
$a^2-a-20=0$
$(a-5)(a+4)=0$
$a=5$
$a=-4$

7 If $t=\tan\frac{\theta}{2}$, which of the following expressions is equivalent to $4\sin\theta+3\cos\theta+5$?

~~(A)~~ $\frac{2(t+2)^2}{1-t^2}$

~~(B)~~ $\frac{(t+4)^2}{1-t^2}$

(C) $\frac{2(t+2)^2}{1+t^2}$

(D) $\frac{(t+4)^2}{1+t^2}$

$4\times\frac{2t}{1+t^2} + 3\times\frac{(1-t^2)}{1+t^2} + 5$

$\frac{8t+3-\cancel{3t^2}+5+\cancel{5t^2}}{1+t^2}$

$\frac{2t^2+8t+8}{1+t^2} = \frac{2(t+2)^2}{1+t^2}$

8 Which of the following is a correct expression for $\tan\left(x+\frac{\pi}{4}\right)$?

(A) $\frac{\cos x+\sin x}{\cos x-\sin x}$

~~(B)~~ $\frac{\cos x+2\sin x}{\cos x-\sin x}$

~~(C)~~ $\frac{\cos x+\sin x}{\cos^2 x-\sin x}$

~~(D)~~ $\frac{\cos x-\sin x}{\cos x-\sin x}$

$\frac{\tan x+1}{1-\tan x} = \frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}$

$=\frac{\sin x+\cos x}{\cos x-\sin x}$

9 The curve $y=2x^{\frac{1}{3}}$ is reflected in the line $y=x$.
What is the equation of the reflected curve?

(A) $y=\frac{x^3}{16}$

(B) $y=\frac{x^3}{8}$

(C) $y=\frac{x^3}{4}$

(D) $y=\frac{x^3}{2}$

$y=2x^{\frac{1}{3}}$
swap x and y
$x=2y^{\frac{1}{3}}$
$\frac{x}{2}=y^{\frac{1}{3}}$
cube
$\frac{x^3}{8}=y$

10 A particle is moving in simple harmonic motion with displacement x.
Its velocity is given by $v^2=9(36-x^2)$.
What is the amplitude, A, of the motion and the maximum speed of the particle?

~~(A)~~ $A=3$ and maximum speed $v=6$
~~(B)~~ $A=3$ and maximum speed $v=18$
(C) $A=6$ and maximum speed $v=18$
(D) $A=6$ and maximum speed $v=6$

End of Section I

$v^2 = 9(36-x^2)$
$= n^2(a^2-x^2)$

$n=3$
$a=6$

$v^2=9\times36$
$v=3\times6$
$=18$

-6 6
min accel.
MAX v.
velocity
MAX accel.

Question (11)

(a) $y = \tan^{-1}(2x^{-1})$

$$\frac{dy}{dx} = \frac{-2x^{-2}}{1+\frac{4}{x^2}} \checkmark$$

$$= \frac{-2}{x^2+4}. \checkmark$$

Comment:

• Well done
• straight forward question if you use $\frac{d}{dx}\tan^{-1}[f(x)] = \frac{f'(x)}{1+[f(x)]^2}$
• or equivalent merit

(b) $$\int_0^{3/2} \frac{dx}{\sqrt{9-4x^2}}$$

$$= \frac{1}{2}\int_0^{3/2} \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}} \checkmark$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{3}\right]_0^{3/2}$$

$$= \pi/4 \checkmark$$

Comment:

• Deduct [1] if common factor 4 is not taken out.
• Allow CFE and ignore subsequent errors. 1 mark (max) for not taken out 4
• Well done for most candidates if

(c) $$\tan 45 = \left|\frac{p-q}{1+pq}\right| \checkmark \quad p, q \text{ explained}$$

$$\therefore 1 = \frac{|p-q|}{|1+pq|} \longrightarrow \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \checkmark$$

Where gradient of tgt at P is p

" " " at Q is q.

Comment:

• Acknowledge (show/prove) gradient of tgt at P and Q is p, q AND correctly use the • Angle between two lines formula. (1)
• $\left|\frac{a}{b}\right| = \frac{\lvert a\rvert}{\lvert b\rvert}$
Have to show what is p, q !

(d) $y = 2\cos^{-1}3x$.

$-1 \le 3x \le 1$

$-\frac{1}{3} \le x \le \frac{1}{3}$ ✓ $\left(\frac{1}{2}\right)$

Comment $0 \le y \le 2\pi$ ✓ $\left(\frac{1}{2}\right)$

Well done
(1 mark) From correctly drawn graph
or explicitly specified domain & range as above

(e) $x^3 - 3x^2 + 4x + 2 = 0$

✓ $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, $\left\{ \begin{array}{l} \sum \alpha_i = 3. \\ \sum_{i \ne j} \alpha_i \alpha_j = 4 \end{array} \right.$ ✓

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{4}{-2}$ and correctly evaluated

Comment 2.

Almost all got this part correct

(f) $4\sin\theta + 3\cos\theta = -5$

$4\sin\theta + 3\cos\theta = r\sin(\theta + \alpha)$ expanding

$r = \sqrt{4^2 + 3^2} = 5$

and $\alpha = 3/4$ (in the 1st quad.

$\alpha = 36°52'$

$5\sin(\theta + 36°52') = -5$

$\sin(\theta + 36°52') = -1$

$\theta + 36°52' = 270°, \ldots$ (contrary

Comment $\doteqdot 233°8'$

Expanding $r\sin(\theta+\alpha)$
$= 4\sin\theta + 3\cos\theta$ ✓
equate $\sin\theta$ & $\cos\theta$
or 't' substitution ✓ ✓
$233°8'$ (3rd & 4th quad)
correct solution band → 1 mark

(g) $y = \dfrac{x}{(x+3)(x+4)}$

(i) $\dfrac{dy}{dx} = \dfrac{(x+3)(x+4) - x(2x+7)}{(x+3)^2(x+4)^2}$

$= \dfrac{-x^2 + 7x - 7x + 12}{(x+3)^2(x+4)^2}$

$\dfrac{dy}{dx} = 0 \implies \dfrac{12 - x^2}{(x+3)^2(x+4)^2} = 0$

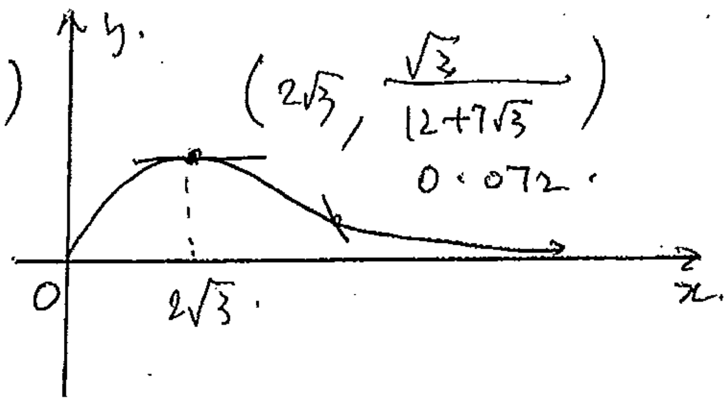
i.e. $12 - x^2 = 0$

$x = \pm\sqrt{12} = \pm 2\sqrt{3}$.

Comment:

Well done this part
Most know how to use quotient rule and obtained two solutions $12 - x^2 = 0$.

(ii)



- passes through (0,0) ✓ $\frac{1}{2}$
- max $\left(2\sqrt{3}, \dfrac{\sqrt{3}}{12+7\sqrt{3}}\right)$ ✓ $\frac{1}{2}$
  and test for max.
- Horizontal asymptote $y=0$ and arrow (to infinity...). ✓ $\frac{1}{2}$
- sign diagram.
  approx markings for pt of inflex ✓ $\frac{1}{2}$

12) (a)(i) $N = P(1-e^{-kt})$

3 Unit Trial HSC 2016.

$N = P - Pe^{-kt}$

$\frac{dN}{dt} = -Pe^{-kt} \times -k$

$= Pke^{-kt}$

$= k(P-N)$ (1)

now $N = P - Pe^{-kt}$

$Pe^{-kt} = P - N$

Generally well answered but some students got very lost in a quite simple proof.
Many students assumed: $Pe^{-kt} = P - N$ without stating it. This made the proof very inconsistent.

(ii) P = initial amount
N = amount that has evaporated

$\frac{1}{4}P = P(1-e^{-kt})$

$\frac{1}{4} = 1 - e^{-kt}$

$e^{-kt} = 1 - \frac{1}{4} = \frac{3}{4}$

$\ln e^{-kt} = \ln\left(\frac{3}{4}\right)$

$-kt = \ln 3 - \ln 4$

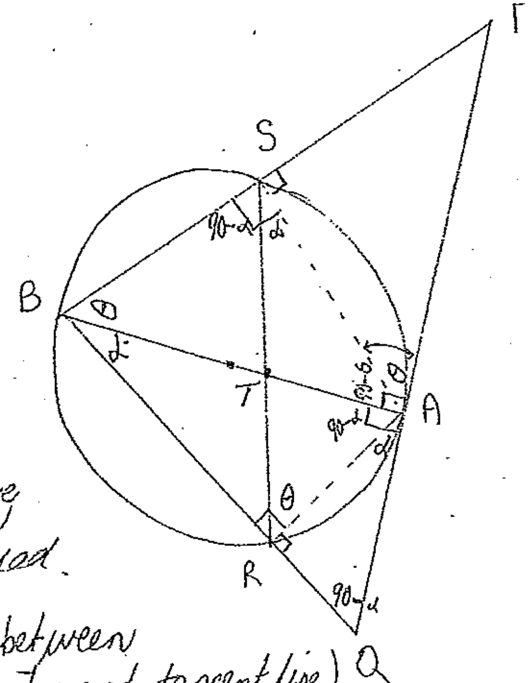
$-kt = \ln 3 - 2\ln 2$

$-t = \frac{\ln 3 - 2\ln 2}{k}$

$t = \frac{-(\ln 3 - 2\ln 2)}{k}$ (2)

Part (ii) well answered by most!

(b)



Generally well done. A few people stated that SR is also a diameter. Not so!
A few methods where available to show PQRS is a cyclic quad.

$B\hat{A}P = 90°$ (line between diameter and tangent line)

$B\hat{R}A = 90°$ (angle in a semi circle)

$S\hat{A}P = A\hat{B}S = \theta$ (alternate segment).

let $R\hat{B}A = \alpha$, $R\hat{S}A = \alpha$ (angles standing on same arc).

$B\hat{S}A = 90°$ (opposite angle cyclic quad. BSAR supplementary angles).

$A\hat{B}S = A\hat{R}T = \theta$ (angles standing on same arc)

Why is PQRS a cyclic quad?

$P\hat{Q}R = 90 - \alpha$, $R\hat{S}P = 90 + \alpha$

so $P\hat{Q}R + R\hat{S}P = 180°$ (opposite angles in cyclic quad add to 180°). (3)

(c) 11 people.
2 circular tables

6, 5

Question badly answered by a large number of students.

$\frac{11!}{5!\,6!} \times \cancel{5!} \times 4!$

$\frac{11!}{30}$

${}^{11}C_6 \times 5! \times 1 \times 4!$

Pick the people. arrange around table. 5 people left. arrange around table.

$= 462 \times 120 \times 24 = 1,330,560$ (2)

(d) $f(x) = 3x - x^3$

$= x(3 - x^2)$

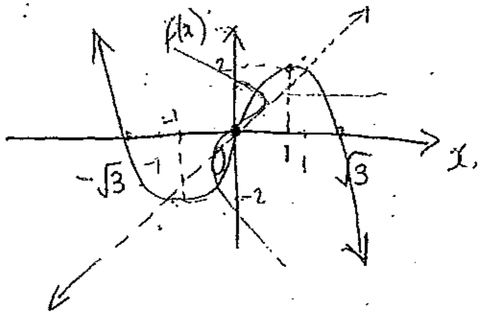
$= x(\sqrt{3} - x)(\sqrt{3} + x)$

$x = 3x - x^3$

$x^3 - 2x = 0$

$x(x^2 - 2) = 0$

$x(x - \sqrt{2})(x + \sqrt{2})$



let $y = 3x - x^3$

$x = 3y - y^3$

$x = y(3 - y^2)$

$= y(\sqrt{3} - y)(\sqrt{3} + y)$

$y' = 3 - 3x^2$

$= 3(1 - x^2)$

$= 3(1 - x)(1 + x)$

$x = 1, -1$

$y = 2, -2$

$y'' = -6x$

(i) $-1 \leq x \leq 1$ (2) Badly answered.

(ii) $-2 \leq x \leq 2$ (1)

Domain of $f^{-1}(x)$ is the range of $f(x)$ in its specified domain.

(e) $\tan 45^\circ = \frac{OT}{OA}$

$OA = OT$

$\tan 30^\circ = \frac{OT}{OB}$

$\frac{OB}{\sqrt{3}} = OT \Rightarrow OB = \sqrt{3}\,OT$

$\triangle AOB$, $(AO)^2 + (BO)^2 = 10000$

$(OT)^2 + 3(OT)^2 = 10000$

$4(OT)^2 = 10000$

$(OT)^2 = 2500$

$OT = 50\,m$

cliff is 50 m.

Well answered by nearly all students. (2)
A bookwork type question + the diagram ~~was included~~ $\Rightarrow$ helps a lot!

(f) $\ddot{x} = -e^{-2x}$

$\int \frac{d}{dx}\left(\frac{1}{2}v^2\right) dx = \int -e^{-2x}\, dx$ $\boxed{\frac{1}{2}}$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + C$

data: $t=0$, $x=0$, $v=1$

$\frac{1}{2} = \frac{1}{2}e^0 + C$

$C = 0$ $\boxed{\frac{1}{2}}$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$

$v^2 = e^{-2x}$

$v = \left(e^{-2x}\right)^{\frac{1}{2}}$

$v = e^{-x}$

$\frac{dx}{dt} = e^{-x} = \frac{1}{e^x}$

$\frac{dt}{dx} = e^x$

$t = e^x + C_1$

data: $t=0$, $x=0$

$0 = 1 + C_1$

$C_1 = -1$ $\boxed{\frac{1}{2}}$

$t = e^x - 1$

so $e^x = t+1$

$\ln e^x = \ln(t+1)$

$x = \ln(t+1)$ $\boxed{\frac{1}{2}}$

Generally well attempted. Using $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ was the key.

total marks 2

13) a) i) $\tan\theta = \frac{x}{AC}$

$AC = \frac{x}{\tan\theta}$

$\tan 2\theta = \frac{x}{OC}$

$OC = \frac{x}{\tan 2\theta}$

$AC + OC = 1$

$\frac{x}{\tan\theta} + \frac{x}{\tan 2\theta} = 1$

$\frac{x}{\tan\theta} + \frac{x}{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)} = 1$

$\frac{2x + x - \tan^2\theta\, x}{2\tan\theta} = 1$

$x(3 - \tan^2\theta) = 2\tan\theta$

$x = \frac{2\tan\theta}{3-\tan^2\theta}$

ii) $\frac{\sqrt{3}}{4} = \frac{2\tan\theta}{3-\tan^2\theta}$

$3\sqrt{3} - \sqrt{3}\tan^2\theta = 8\tan\theta$

$\sqrt{3}\tan^2\theta + 8\tan\theta - 3\sqrt{3} = 0$

$\tan\theta = \frac{-(8) \pm \sqrt{(8)^2 - 4(\sqrt{3})(-3\sqrt{3})}}{2(\sqrt{3})}$

$= \frac{-8 \pm 10}{2\sqrt{3}}$

$= \frac{-4 \pm 5}{\sqrt{3}}$

$$\tan\theta = \frac{1}{\sqrt{3}} \quad \text{or} \quad -\frac{9}{\sqrt{3}}$$

$$\theta = 30^\circ \quad (\text{since } \theta \text{ is acute})$$

$$2\theta = 60^\circ$$

$$A = \tfrac{1}{2}(1)(1)\sin 60^\circ$$
$$= \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}}{4} \text{ square metres.}$$

COMMENT:
Part (i) was done reasonably well.
Many students failed to recognise that there was a quadratic in $\tan\theta$ which could be solved to find θ.

b) $f(x) = \frac{1}{x} - a$
$f(x) = x^{-1} - a$
$f'(x) = -x^{-2}$
$= -\frac{1}{x^2}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{\frac{1}{x_1} - a}{-\frac{1}{x_1^2}} \times \frac{-x_1^2}{-x_1^2}$$

$$x_2 = x_1 + x_1 - ax_1^2$$

$$x_2 = 2x_1 - ax_1^2$$

$$x_2 = x_1(2 - ax_1)$$

COMMENT:
A different style of question on first impressions. However, it is just a simple application of Newton's method.

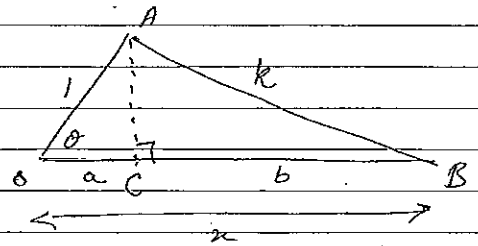
c) $\displaystyle \lim_{x\to 0} \frac{\sin 2x}{\tan 3x}$

$$= \lim_{x\to 0} \frac{\sin 2x}{2x} \times \frac{3x}{\tan 3x} \times \frac{2}{3}$$

$$= 1 \times 1 \times \frac{2}{3}$$

$$= \frac{2}{3}$$

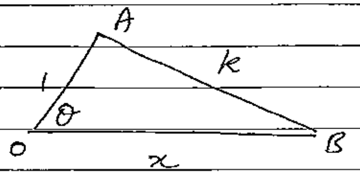
d) i)



$\cos\theta = \frac{a}{1}$ $\quad \sin\theta = \frac{AC}{1}$

$$x = a + b$$
$$x = \cos\theta + \sqrt{k^2 - \sin^2\theta}$$

OR



$$\cos\theta = \frac{1^2 + x^2 - k^2}{2(1)(x)}$$

$$2x\cos\theta = 1 + x^2 - k^2$$

$$x^2 - 2\cos\theta\, x + 1 - k^2 = 0$$

$$x^2 - 2\cos\theta\, x + \cos^2\theta + \sin^2\theta - k^2 = 0$$

$$(x - \cos\theta)^2 = k^2 - \sin^2\theta$$

$$x - \cos\theta = \pm\sqrt{k^2 - \sin^2\theta}$$

$$x = \cos\theta \pm \sqrt{k^2 - \sin^2\theta}$$

$$x = \cos\theta + \sqrt{k^2 - \sin^2\theta} \quad \text{metres.}$$

since x is a distance

COMMENT:
Students that assumed AB was a tangent could not get the result.

ii) $x = \cos\theta + \sqrt{k^2 - \sin^2\theta}$

$$x = \cos\theta + (k^2 - \sin^2\theta)^{\frac{1}{2}}$$

$$\frac{dx}{d\theta} = -\sin\theta + \frac{1}{2}(k^2 - \sin^2\theta)^{-\frac{1}{2}} \cdot (-2\sin\theta\cos\theta)$$

$$\frac{dx}{d\theta} = -\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}}\right) \quad \cancel{\text{m/rad.}}$$

iii) $\dfrac{dx}{dt} = \dfrac{dx}{d\theta} \times \dfrac{d\theta}{dt}$

$$\frac{dx}{dt} = -\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}}\right) \times 4\pi$$

when $\theta = \frac{\pi}{6}$.

$$\frac{dx}{dt} = -\sin\frac{\pi}{6}\left(1 + \frac{\cos\frac{\pi}{6}}{\sqrt{k^2 - (\sin\frac{\pi}{6})^2}}\right)4\pi$$

$$= -\left(\frac{1}{2}\right)\left(1 + \frac{\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{k^2 - \left(\frac{1}{2}\right)^2}}\right)4\pi$$

$$= -2\pi\left(1 + \frac{\sqrt{3}}{2\sqrt{k^2 - \frac{1}{4}}}\right)$$

$$= -2\pi\left(1 + \frac{\sqrt{3}}{\sqrt{4k^2 - 1}}\right) \quad \text{m/s.}$$

COMMENT:
This answer could be written a number of different ways.

iv) considering the scenario, the point B will change direction when $\theta = 0, \pi$

From the equation $\dfrac{dx}{dt} = 0$ when $\dfrac{dx}{d\theta} = 0$

since $\dfrac{d\theta}{dt}$ is a constant.

$$-\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}}\right) = 0$$

$\sin\theta = 0$ or $\frac{1+\cos\theta}{\sqrt{k^2-\sin^2\theta}} = 0$

$\theta = 0, \pi$

$\sqrt{k^2-\sin^2\theta} + \cos\theta = 0$

$\sqrt{k^2-\sin^2\theta} = -\cos\theta$

$k^2-\sin^2\theta = \cos^2\theta$

$k^2 = \sin^2\theta + \cos^2\theta$

$k^2 = 1$

since $k > 1$

no solution.

COMMENT:

Students should not be using the equation which has $\theta = \frac{\pi}{6}$ substituted.

This should have been an easy mark

QUESTION 14. (X1)

(a) Aim To prove $\sum_{r=1}^{n}(r+3)2^r = (n+2)2^{n+1} - 4$ for $n \in Z^+$

Step I when $n = 1$

LHS $= (1+3)2 = 8$ RHS $= 3 \times 4 - 4 = 8$

$\therefore$ true when $n = 1$.

Step II Assume $\sum_{r=1}^{k}(r+3)2^r = (k+2)2^{k+1} - 4$.

Step III Assuming step II is true

Prove true for $n = k+1$.

ie. $\sum_{r=1}^{k+1}(r+3)2^r = (k+3)2^{k+2} - 4$.

[3]

now LHS $= \sum_{r=1}^{k}(r+3)2^k + (k+4)2^{k+1}$

$= (k+2)2^{k+1} - 4 + (k+4)2^{k+1}$

$= (2k+6)2^{k+1} - 4$

$= 2(k+3)2^{k+1} - 4$

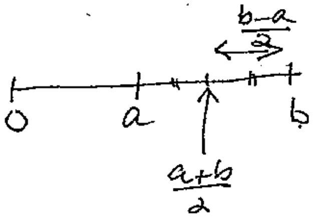
$= (k+3)2^{k+2} - 4$

$=$ RHS

Step IV We conclude that by the Principle of Mathematical Induction the statement is true for $n \in Z^+$

COMMENT: This was a straightforward question on Induction and was well done. Most scored full marks.

(b) (i)



[1]

The amplitude is $\frac{b-a}{2}$.

COMMENT: The common error was $\frac{a+b}{2}$ which is the centre of motion.

(ii) Using $v^2 = n^2\left[\left(\frac{b-a}{2}\right)^2 - \left(x - \left(\frac{a+b}{2}\right)\right)^2\right]$

we have $v_{MAX}^2 = n^2\left(\frac{b-a}{2}\right)^2$

ie. $v_{max} = n\frac{(b-a)}{2}$

[3]

$\therefore V = \frac{n(b-a)}{2}$

$n = \frac{2V}{n(b-a)}$

Hence $T = \frac{2\pi}{n}$

$= \frac{2\pi}{2V/n(b-a)}$

$= \frac{\pi(b-a)}{V}$

COMMENT: not well done. As is often the case where the answer is provided there was a tendency to contrive the answer using a circular argument.

(c) (i) given $x = Vt\cos\alpha$ & $y = Vt\sin\alpha - \frac{1}{2}gt^2$.

$t = \frac{x}{V\cos\alpha}$

$\therefore y = V\frac{x}{V\cos\alpha}\sin\alpha - \frac{1}{2}\frac{g}{V^2}\frac{x^2}{\cos^2\alpha}$

ie. $y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{V^2}\sec^2\alpha$ (A)

& $V = \sqrt{2gh}$ (given)

$\therefore V^2 = 2gh$

$\therefore$ (A) becomes

$y = x\tan\alpha - \frac{1}{2}\frac{g x^2}{2gh}\sec^2\alpha$

OR

$y = x\tan\alpha - \frac{x^2}{4h}(1+\tan^2\alpha)$ (B)

To find P

we solve (B) and $y = \frac{x^2}{4a}$ (C) i.e. $\frac{x^2}{4a} = x\tan\alpha - \frac{x^2}{4h}(1+\tan^2\alpha)$

$$x^2\left[\frac{1+\tan^2\alpha}{4h} + \frac{1}{4a}\right] - x\tan\alpha = 0$$

$$x\left[\left(\frac{1+\tan^2\alpha}{4h} + \frac{1}{4a}\right)x - \tan\alpha\right] = 0$$

$\therefore x = 0$ or $x = \dfrac{\tan\alpha}{\frac{1+\tan^2\alpha}{4h} + \frac{1}{4a}}$ [3]

$\uparrow$

$x \neq 0.$

$$= \frac{4ah\tan\alpha}{a(1+\tan^2\alpha)+h}$$

$$= \frac{4ah\tan\alpha}{a+h+a\tan^2\alpha}$$

$$= \frac{4ah}{\frac{a+h}{\tan\alpha} + a\tan\alpha}$$

$$= \frac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$$

COMMENT most realised to solve (B) and (C). Unfortunately not many were able to do so successfully.

(ii). Given $f(\theta) = (a+h)\cot\theta + a\tan\theta$ for $0<\theta<\frac{\pi}{2}$

$$f'(\theta) = (a+h)\times -\csc^2\theta + a\sec^2\theta.$$

For st. point

$f'(\theta) = 0$

$\therefore a\sec^2\theta = (a+h)\csc^2\theta$ [2]

$$\frac{a}{\cos^2\theta} = \frac{(a+h)}{\sin^2\theta}$$

$$\therefore \frac{\sin^2\theta}{\cos^2\theta} = \frac{a+h}{a}$$

$$\tan^2\theta = \frac{a+h}{a}$$

$$\tan\theta = \pm\sqrt{\frac{a+h}{a}}$$

$\therefore \tan\theta = \sqrt{\frac{a+h}{a}}$ (as $\tan\theta \neq$ negative value since $0<\theta<\frac{\pi}{2}$)

Clearly this st. point is a minimum since $f(\theta) \to \infty$ as $\theta \to 0$ [$\cot\theta \to \infty$]

& $f(\theta) \to \infty$ as $\theta \to \frac{\pi}{2}$. [$\tan\theta \to \infty$]

COMMENT Some students failed to justify the positive value for $\tan\theta$ and lost $\frac{1}{2}$ mark.
It was possible to show that $f''(\theta) > 0$
$\therefore$ MIN.

(iii) given $x = \dfrac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$. (D)

the max. value will occur

when $(a+h)\cot\alpha + a\tan\alpha$

is least. ie. when $\tan\alpha = \sqrt{\frac{a+h}{a}}$

ie. in D

$$x_{MAX} = \frac{4ah}{(a+h)\sqrt{\frac{a}{a+h}} + a\sqrt{\frac{a+h}{a}}}$$ [3]

$$= \frac{4ah}{\sqrt{a(a+h)} + \sqrt{a(a+h)}}$$

$$= \frac{4ah}{2\sqrt{a(a+h)}}$$

$$= 2h\sqrt{\frac{a}{a+h}}$$

as required

COMMENT: ~~most students~~ recognised the connection with parts (i) & (ii).

The question proved easy for the majority of ~~most~~ students