2016 SYDNEY BOYS HIGH SCHOOL
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 3–7

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 8–19

90 marks

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Examiner: E.C.

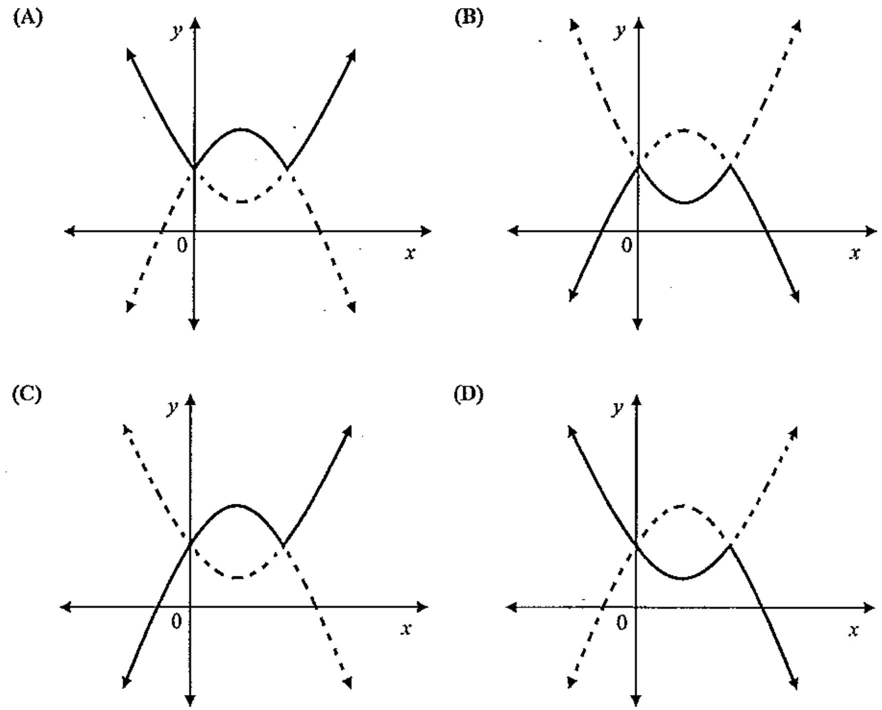
Section I

10 marks
Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1. Which of the following figures in solid line represents $y = 1 - |x - x^2|$?



2. In how many ways can eight students be divided into 2 groups of four, for a polo match?

(A) 70

(B) 140

(C) 35

(D) 50

3. Find $\int \frac{dx}{x^2+2x+2}$.

(A) $\tan^{-1}(x+2)+C$

(B) $\tan^{-1}(x+1)+C$

(C) $\sin^{-1}(x+1)+C$

(D) $\cos^{-1}(x+1)+C$

4. The linear factors of $z^2+6z+10$ over the complex field are:

(A) $(z+3+i)(z-3+i)$

(B) $(z+3+i)^2$

(C) $(z+3-i)(z+3+i)$

(D) $(z+3+i)(z-3-i)$

5. The gradient of the tangent to the curve $\sin x+2\sin y=1$ at the point $\left(\pi,\frac{\pi}{6}\right)$ is:

(A) $\frac{1}{\sqrt{3}}$

(B) $-\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $-\sqrt{3}$

6. The base of a solid is a circle $x^2+y^2=16$. Every cross section of the solid taken perpendicular to the x-axis is a right-angled isosceles triangle with its hypotenuse lying in the base of the solid.

Which of the following is an expression for the volume of the solid?

(A) $\frac{1}{4}\int_{-4}^{4}\left(16-x^2\right)\,dx$

(B) $\int_{-4}^{4}\left(16-x^2\right)\,dx$

(C) $2\int_{-4}^{4}\left(16-x^2\right)\,dx$

(D) $4\int_{-4}^{4}\left(16-x^2\right)\,dx$

7. A particle of mass 1 kg is projected vertically upwards from level with a velocity u m/s. The particle is subject to a constant gravitational force and a resistance which is proportional to the square of its velocity v m/s, (with k being the constant of proportionality).

Let x be the displacement in metres from the ground after t seconds and let g be the acceleration due to gravity.

Which of the following expressions gives the maximum height reached by the particle?

(A) $\displaystyle\int_u^0 \frac{v}{g+kv^2}\,dv$

(B) $\displaystyle\int_u^0 \frac{v}{g-kv^2}\,dv$

(C) $\displaystyle\int_0^u \frac{v}{g+kv^2}\,dv$

(D) $\displaystyle\int_0^u \frac{v}{g-kv^2}\,dv$

8. The equation $x^4+px+q=0$, where $p\neq 0$ and $q\neq 0$, has roots α, β, γ and δ. What is the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$?

(A) $-4q$

(B) p^2-2q

(C) p^4-2q

(D) p^4

9. The solutions to the equation $x^4+4x^3+6x^2-4x+1=0$ are $x=\alpha,\beta,\gamma,$ and δ.

What is the value of $\alpha^2+\beta^2+\gamma^2+\delta^2$?

(A) 4

(B) 16

(C) 28

(D) 32

10. Consider the integral $\displaystyle\int_{-b}^{b} f\left(a-\frac{x}{b}\right)dx$, where a and b are constants. Which of the following integrals is equal to this integral.

(A) $\displaystyle -b\int_{a-1}^{a+1} f(x)\,dx$

(B) $\displaystyle b\int_{a-1}^{a+1} f(x)\,dx$

(C) $\displaystyle -\frac{1}{b}\int_{a-1}^{a+1} f(x)\,dx$

(D) $\displaystyle \frac{1}{b}\int_{a-1}^{a+1} f(x)\,dx$

Section II

90 marks
Attempt Questions 11–16

Allow about 2 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Given that $z = w + \frac{1}{w}$, where $w = 2(\cos\theta + i\sin\theta)$,

(i) Express the real and imaginary parts of z in terms of θ. **2**

(ii) Show that the point representing z in the Argand diagram lies on the curve with Cartesian equation $\frac{x^2}{25} + \frac{y^2}{9} = \frac{1}{4}$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4 + 5\cos x}\, dx$ using the substitution $t = \tan\left(\frac{x}{2}\right)$. **4**

(c) The point P in the Argand diagram represents the variable complex number Z and the point Q is in the first quadrant represent the complex number w, where $w = 1 + 3i$.

Sketch, on separate diagrams, the locus of P in each of the following cases making clear the relationship between the locus and the point Q.

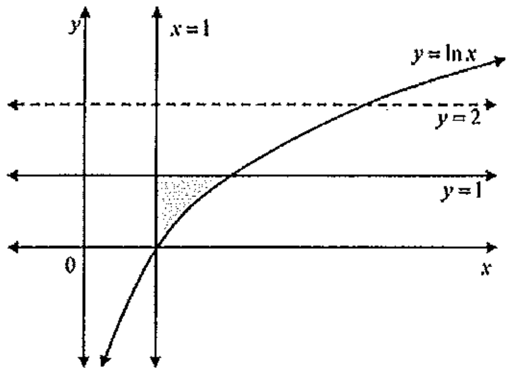
(i) $|z| = |w|$ **1**

(ii) $|z - w| = 2|w|$ **1**

(iii) $|z - w| = |z|$ **1**

Question 11 continues on page 9

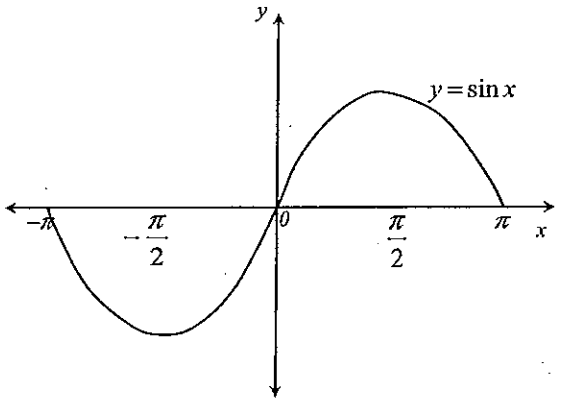
(d) The region bounded by the curve $y = \ln x$, $x = 1$ and $y = 1$ is shaded in the diagram below. The region is rotated about the line $y = 2$ to form a solid. Using the method of cylindrical shells, find the volume of the solid formed. **4**



End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $f(x)=\sin x$ is shown below for the interval $-\pi \le x \le \pi$.



Draw a separate half page graph for each of the following functions.

(i) $y=\dfrac{1}{f\left(x-\frac{\pi}{2}\right)}$ **1**

(ii) $y=f\left(\sqrt{|x|}\right)$ **2**

(b) Let $P(z)=z^5-1$ and $\alpha=\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)$.

(i) Show that 1, α, $\frac{1}{\alpha}$, α^2 and $\frac{1}{\alpha^2}$ are roots of the equation $P(z)=0$. **1**

(ii) Prove the identity $z^5-1=z^2(z-1)\left[\left(z+\frac{1}{z}\right)^2+\left(z+\frac{1}{z}\right)-1\right]$. **1**

(iii) Using the results of (i) and (ii), show that $4\cos^2\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{2\pi}{5}\right)-1=0$. **2**

(iv) Hence, or otherwise, show that $\cos\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}-1}{4}$. **1**

(c) How many diagonals does a regular undecagon (11-sided polygon) have? **2**

Question 12 continues on page 11

(d) Let $y=x^{n-1}\left(1+x^2\right)^{\frac{1}{2}}$.

(i) Where n is a positive integer. Find $\frac{dy}{dx}$. **1**

(ii) Let $I_n=\int \frac{x^n}{\sqrt{1+x^2}}\,dx$, where n is an integer and $n\ge 0$. Using the result from (i), show **2**

that $I_n+\frac{n-1}{n}I_{n-2}=\frac{x^{n-1}\left(1+x^2\right)^{\frac{1}{2}}}{n}$.

(iii) Hence, find $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$. **2**

End of Question 12

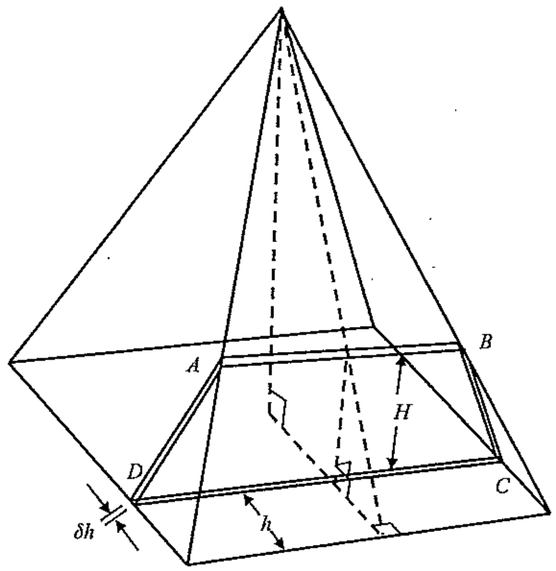
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) n people are sitting around a table.

(i) If p is the probability that two particular men A, B are sitting next to each other, find p. **2**

(ii) If q is the probability that three particular men A, B, C are sitting in a group, find q. **2**

(b) A square pyramid of height 24 cm on a base of side 12 cm is drawn below.



Slicing the square pyramid perpendicular to its base and h units away from one of the edge gives an isosceles trapezium slice $ABCD$, where $CD = 12$ cm and thickness δh as shown on the diagram. Let H be the perpendicular distance between the parallel side AB and CD of the trapezium $ABCD$.

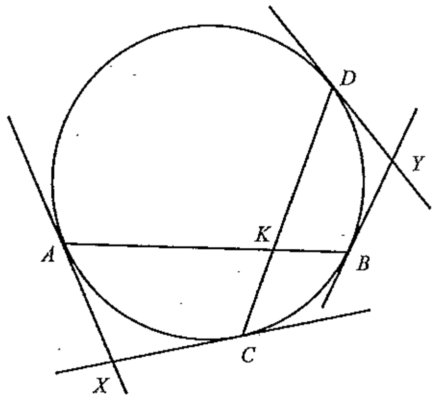
(i) Let AB be the length of the shorter parallel side of the trapezium $ABCD$. Show that $AB = 12 - 2h$ and $H = 4h$. **2**

(ii) Hence, show that the volume of the square pyramid by the method of slicing is given by **2**

$$V = 2\int_0^6 4h(12-h)\ dh.$$

Question 13 continues on page 13

(c) Two chords AKB, CKD of a circle cut at K. The tangents at A and C meet at X, the tangents at B and D meet at Y. **3**



Prove that $\angle AXC + \angle BYD = 2\angle AKD$.

(d) Show that $x^3 - x + 2 = 0$ cannot have a double root. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) If $2\sin 2x+\cos 2x=k$, where k is a real constant and $k\neq -1$. **2**
Show that $(1+k)\tan^2 x-4\tan x+(k-1)=0$.

(ii) Hence, show that if $\tan x_1$, and $\tan x_2$ are roots of this quadratics equation in $\tan x$, then $\tan(x_1+x_2)=2$. **2**

(b) Let a_r be the coefficient of x^r in the expansion of $\left(1+x+x^2\right)^n$.

(i) Show that $a_0+a_2+a_4+...+a_{2n}=\dfrac{3^n+1}{2}$. **2**

(ii) Show that $a_r=a_{2n-r}$. **2**

(c) Let z be the complex number $z=r(\cos\theta+i\sin\theta)$.

(i) Show that $\dfrac{z^2}{\bar{z}}=r(\cos 3\theta+i\sin 3\theta)$. **1**

(ii) If $z^2=i\bar{z}$, find the value of r and the three possible values of θ. **1**

(iii) If $w=\cos\alpha+i(1+\sin\alpha)$, where $-\pi<\alpha\leq\pi$ find the values of $|w-i|$. **2**

(iv) Using (ii), solve the equation $(w-i)^2+1=i\bar{w}$, giving your answers in Cartesian form, **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

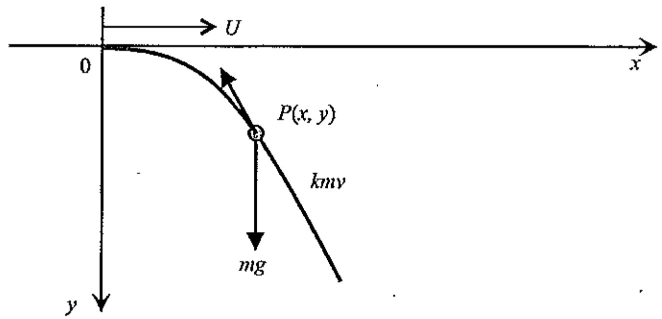
(a) Given that $f(x)=xe^{-x}$, prove by mathematical induction that $f^{(n)}(x)=(-1)^n e^{-x}(x-n)$ for all positive integers n, where $f^{(n)}(x)$ is the nth derivative of $f(x)=xe^{-x}$. **4**

(b) A bomb P of mass m is released from rest by a stealth bomber flying horizontally at a speed U. The bomb experiences the effect of gravity, and a resistance proportional to its velocity v in both the horizontal and vertical direction at any time t, where v is the speed of the bomb at time t. From the diagram, the equations of motion in the horizontal and the vertical directions are given respectively by

$$\ddot{x}=-k\dot{x} \text{ and } \ddot{y}=g-k\dot{y}$$

(Do **NOT** prove this).

where k is a constant and the acceleration due to gravity is g.



Note that the downwards direction is positive.

(i) Show that, at time t after the release, the bomb has travelled a horizontal distance **3**

$$\frac{U\left(1-e^{-kt}\right)}{k} \text{ metres.}$$

(ii) Show that, at time t after the release, the bomb is inclined at an angle **3**

$$\tan^{-1}\left[\frac{g\left(e^{kt}-1\right)}{kU}\right]$$

to the horizontal.

(iii) By considering the components of the velocity of the bomb, show that terminal velocity, V, of the bomb is **2**

$$V=\frac{g}{k}.$$

(iv) Show that the least speed of the bomb is W where $W^2=\dfrac{U^2V^2}{U^2+V^2}$ for $t\geq 0$. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $2\cos A\sin B=\sin(A+B)-\sin(A-B)$, by expanding the right hand side. **1**

(ii) The distinct points $P(\theta,\sin\theta)$ and $Q(\phi,\sin\phi)$ lie on the curve $y=\sin x$, where x is measured in radians. Show the gradient of the chord PQ may be expressed as **2**

$$\frac{\sin\left(\frac{\phi-\theta}{2}\right)\cos\left(\frac{\phi+\theta}{2}\right)}{\left(\frac{\phi-\theta}{2}\right)}.$$

Deduce that if ϕ is approximately equal to θ then the gradient of gradient PQ is approximately equal to $\cos\theta$.

(b) Ten people arrived in the Kingsmith Airport from London. It's late at night, only 4 immigration counters are open. In how many ways can 10 people line up in a 4-lane queue? **2**

Question 16 continues on page 19

(c) Let n be a positive integer.

(i) Show that $\frac{1}{1-t^2}=\left(1+t^2+t^4+\ldots+t^{2n-2}\right)+\frac{t^{2n}}{1-t^2}$ for $t^2\neq 1$. **2**

(ii) For $-1<x<1$, show that $\int_0^x \frac{t}{1-t^2}\,dt=\ln\left(\frac{1}{\sqrt{1-x^2}}\right)$. **2**

(iii) Using the above parts and by letting $x=\sqrt{\frac{8}{9}}$, deduce that **3**

$$\int_0^{\sqrt{\frac{8}{9}}}\frac{t^{2n+1}}{1-t^2}\,dt=\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k.$$

(iv) It can be shown that for $0\le t\le\sqrt{\frac{8}{9}}$, $\frac{t^{2n+1}}{1-t^2}\ge 0$ and $\frac{t^{2n+1}}{1-t^2}\le\frac{t^{2n+1}}{1-\frac{8}{9}}$. **3**

(Do NOT prove this)

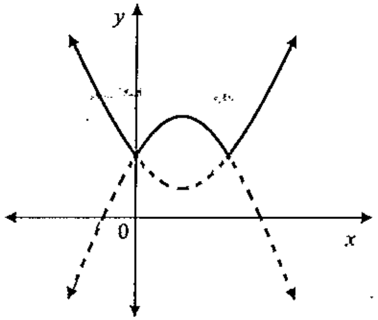
Show that $0\le\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k\le\frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}$.

End of paper

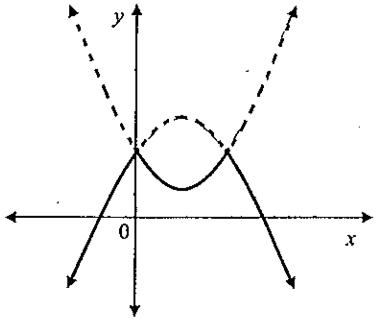
Multiple Choice SOLUTIONS

1. Which of the following figures in solid line represents $y = 1 - |x - x^2|$?

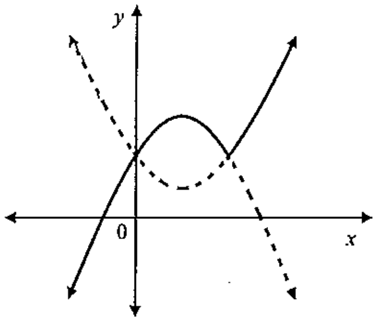
(A)



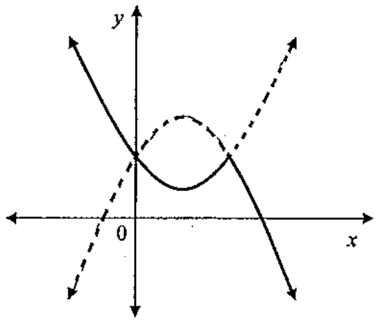
(B)



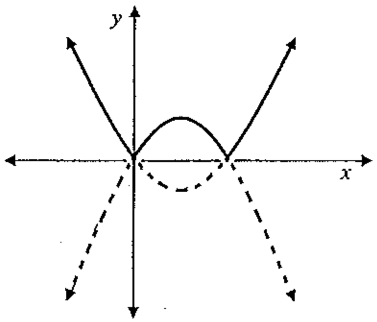
(C)



(D)



The graph of $y = |x - x^2|$ is



So $y = 1 - |x - x^2|$ reflects it in the x-axis and a translation of 1 unit upwards.

2. In how many ways can eight students be divided into 2 groups of four, for a polo match?

(A) 70

(B) 140

(C) 35

(D) 50

${}^8C_4$ is the number of ways to select 4 students, e.g. ABCD, to play against EFGH. But this also accounts for EFGH being selected to play against ABCD. Therefore there are ${}^8C_4 \div 2 = 35$ ways.

3. Find $\int \frac{dx}{x^2 + 2x + 2}$.

(A) $\tan^{-1}(x+2) + C$

(B) $\tan^{-1}(x+1) + C$

(C) $\sin^{-1}(x+1) + C$

(D) $\cos^{-1}(x+1) + C$

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{(x+1)^2 + 1}$$
$$= \tan^{-1}(x+1) + C$$

4. What are the linear factors of $z^2 + 6z + 10$ over the complex field?

(A) $(z + 3 + i)(z - 3 + i)$

(B) $(z + 3 + i)^2$

(C) $(z + 3 - i)(z + 3 + i)$

(D) $(z + 3 + i)(z - 3 - i)$

$$z^2 + 6z + 10 = z^2 + 6z + 9 + 1$$
$$= (z+3)^2 - i^2$$

5. What is the gradient of the tangent to the curve $\sin x + 2\sin y = 1$ at the point $\left(\pi, \frac{\pi}{6}\right)$?

(A) $\frac{1}{\sqrt{3}}$

(B) $-\frac{1}{\sqrt{3}}$

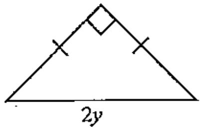
(C) $\sqrt{3}$

(D) $-\sqrt{3}$

$\cos x + 2\cos y . y' = 0$
$\therefore \cos\pi + 2\cos\frac{\pi}{6} . y' = 0$
$\therefore \sqrt{3} . y' = 1$

6. The base of a solid is a circle $x^2 + y^2 = 16$. Every cross section of the solid taken perpendicular to the x-axis is a right-angled isosceles triangle with its hypotenuse lying in the base of the solid.

Which of the following is an expression for the volume of the solid?



(A) $\frac{1}{4}\int_{-4}^{4}\left(16-x^2\right)\,dx$

(B) $\int_{-4}^{4}\left(16-x^2\right)\,dx$

(C) $2\int_{-4}^{4}\left(16-x^2\right)\,dx$

(D) $4\int_{-4}^{4}\left(16-x^2\right)\,dx$

The area of the isosceles right triangle with hypotenuse $2y$ is y^2.
i.e. $16-x^2$

7. A particle of mass 1 kg is projected vertically upwards from level with a velocity u m/s. The particle is subject to a constant gravitational force and a resistance which is proportional to the square of its velocity v m/s, (with k being the constant of proportionality).

Let x be the displacement in metres from the ground after t seconds and let g be the acceleration due to gravity.

Which of the following expressions gives the maximum height reached by the particle?

(A) $\int_{u}^{0}\frac{v}{g+kv^2}\,dv$

(B) $\int_{u}^{0}\frac{v}{g-kv^2}\,dv$

(C) $\int_{0}^{u}\frac{v}{g+kv^2}\,dv$

(D) $\int_{0}^{u}\frac{v}{g-kv^2}\,dv$

Let D be the maximum height.
$ma = -mg - mkv^2$
$\therefore v\frac{dv}{dy} = -(g+kv^2) \Rightarrow \frac{v\,dv}{g+kv^2} = -dy$
$\int_{u}^{0}\frac{v\,dv}{g+kv^2} = -\int_{0}^{D}dy$

8. The equation $x^4 + px + q = 0$, where $p \neq 0$ and $q \neq 0$, has roots α, β, γ and δ. What is the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$?

(A) $-4q$

(B) $p^2 - 2q$

(C) $p^4 - 2q$

(D) p^4

Substitute α, β, γ and δ into $x^4 + px + q = 0$
$\Sigma\alpha = 0$
$$\left.\begin{aligned}\alpha^4+\alpha p+q&=0\\ \beta^4+\beta p+q&=0\\ \gamma^4+\gamma p+q&=0\\ \delta^4+\delta p+q&=0\end{aligned}\right\}+$$
$\Sigma\alpha^4 + p\Sigma\alpha + 4q = 0$
$\therefore \Sigma\alpha^4 = -4q$

9. The solutions to the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ are

$$x = \tan\left(\frac{\pi}{16}\right), \tan\left(\frac{5\pi}{16}\right), \tan\left(\frac{-3\pi}{16}\right), \tan\left(\frac{-7\pi}{16}\right)$$

What is the value of $\tan^2\left(\frac{\pi}{16}\right)+\tan^2\left(\frac{3\pi}{16}\right)+\tan^2\left(\frac{5\pi}{16}\right)+\tan^2\left(\frac{7\pi}{16}\right)$?

(A) 4

(B) 16

(C) 28

(D) 32

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2+\delta^2 &= (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)\\ &= (-4)^2 - 2(-6)\\ &= 28\end{aligned}$$

10. Consider the integral $\int_{-b}^{b} f\left(a-\frac{x}{b}\right) dx$, where a and b are constants.

Which of the following integrals is equal to this integral.

(A) $-b\int_{a-1}^{a+1} f(x)\ dx$

(B) $b\int_{a-1}^{a+1} f(x)\ dx$

(C) $-\frac{1}{b}\int_{a-1}^{a+1} f(x)\ dx$

(D) $\frac{1}{b}\int_{a-1}^{a+1} f(x)\ dx$

Let $u=a-\frac{x}{b}$

$\therefore du=-\frac{1}{b}dx \Rightarrow dx=-b\,du$

$x:-b\sim b$

$u:a+1\sim a-1$

$$\int_{-b}^{b} f\left(a-\frac{x}{b}\right)dx=-b\int_{a+1}^{a-1} f(u)\,du$$

$$=b\int_{a-1}^{a+1} f(u)\,du$$

Question 11 — SOLUTIONS

(a) Given that, $z=w+\frac{1}{w}$, where $w=2(\cos\theta+i\sin\theta)$

(i) Express the real and imaginary parts of z in terms of θ. **2**

$$\begin{aligned} z&=w+\frac{1}{w}\\ &=w+\frac{\overline{w}}{|w|^2}\\ &=2(\cos\theta+i\sin\theta)+\frac{2(\cos\theta-i\sin\theta)}{4}\\ &=\frac{5}{2}\cos\theta+\frac{3}{2}i\sin\theta \end{aligned}$$

$\therefore \operatorname{Re} z=\frac{5}{2}\cos\theta$ and $\operatorname{Im} z=\frac{3}{2}\sin\theta$

Comment

Generally well done, though many people didn't know that $\frac{1}{w}=\frac{1}{2}\operatorname{cis}(-\theta)$.

Much time was wasted by making $w+\frac{1}{w}=\frac{w^2+1}{w}$.

(ii) Show that the point representing z in the Argand diagram lies on the curve with Cartesian equation $\frac{x^2}{25}+\frac{y^2}{9}=\frac{1}{4}$. **2**

Let $x=\frac{5}{2}\cos\theta$ and $y=\frac{3}{2}\sin\theta$

$\therefore \frac{4x^2}{25}=\cos^2\theta$ and $\frac{4y^2}{9}=\sin^2\theta$

$\therefore \frac{4x^2}{25}+\frac{4y^2}{9}=1$

$\therefore \frac{x^2}{25}+\frac{y^2}{9}=\frac{1}{4}$

Comment

The setting out was very poor with this question. Not much detail was presented as proof. Some students did assume the result and this was penalised.

(b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\, dx$ using the substitution $t=\tan\left(\frac{x}{2}\right)$. **4**

$$4+5\cos x = 4+5\times\frac{1-t^2}{1+t^2}$$
$$=\frac{4+4t^2+5-4t^2}{1+t^2}$$
$$=\frac{9-t^2}{1+t^2}$$

x: $0 \sim \frac{\pi}{2}$

t: $0 \sim 1$

$$\frac{1}{4+5\cos x}=\frac{1+t^2}{9-t^2}$$

$$t=\tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} t$$
$$\therefore dx = \frac{2dt}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\cos x} = \int_0^1 \frac{1+t^2}{9-t^2}\times\frac{2dt}{1+t^2}$$
$$=\int_0^1 \frac{2dt}{9-t^2}$$
$$=\int_0^1 \frac{2dt}{(3-t)(3+t)}$$
$$=\int_0^1 \left(\frac{A}{3-t}+\frac{B}{3+t}\right)dt$$
$$=\int_0^1 \left(\frac{\frac{1}{3}}{3-t}+\frac{\frac{1}{3}}{3+t}\right)dt$$

[See next page for methods]

$$=\frac{1}{3}\left[-\ln|3-t|+\ln|3+t|\right]_0^1$$
$$=\frac{1}{3}\left[\ln\left|\frac{3+t}{3-t}\right|\right]_0^1$$
$$=\frac{1}{3}(\ln 2-\ln 1)$$
$$=\frac{1}{3}\ln 2$$

Comment

Too many students don't know how to get to $dx=\frac{2dt}{1+t^2}$ quickly.

Step 1 – memorise it! or Step 2 – rewrite $t=\tan\frac{x}{2}$ as $x=2\tan^{-1}t$.

Cover up method

$$\frac{2}{(3-t)(3+t)}=\frac{A}{3-t}+\frac{B}{3+t}$$

To find A: Substitute $t=3$ into $\frac{2}{\cancel{(3-t)}(3+t)}$

$$\therefore A=\frac{1}{3}$$

To find B: Substitute $t=-3$ into $\frac{2}{(3-t)\cancel{(3+t)}}$

$$\therefore B=\frac{1}{3}$$

Identical Polynomials Method

$$\frac{2}{(3-t)(3+t)}=\frac{A}{3-t}+\frac{B}{3+t} \quad\Rightarrow\quad 2\equiv A(3+t)+B(3-t)$$

Substitute $t=3$: $2=A(3+3)$

$$\therefore A=\frac{1}{3}$$

Substitute $t=-3$: $2=B(3-(-3))$

$$\therefore B=\frac{1}{3}$$

Looker's Theorem

Given the basic nature of the partial fraction, <u>this type</u> could be done by inspection.

$$\therefore \frac{2}{(3-t)(3+t)}=\frac{\frac{1}{3}}{3-t}+\frac{\frac{1}{3}}{3+t}$$

(c) The point P in the Argand diagram represents the variable complex number Z and the point Q is in the first quadrant represent the complex number w, where $w = 1 + 3i$.

Sketch, on separate diagrams, the locus of P in each of the following cases making clear the relationship between the locus and the point Q.

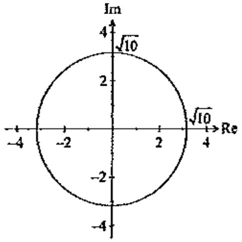
$w = 1 + 3i$

$\therefore |w| = \sqrt{10}$

(i) $|z| = |w|$ **1**

$\therefore |z| = \sqrt{10}$

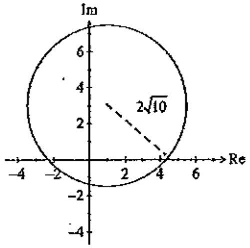
This is a circle with centre the origin and the radius equal to the modulus of w.



Comment:
Generally well done, though some students were confused about the radius

(ii) $|z - w| = 2|w|$ **1**

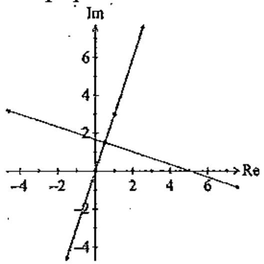
This is a circle with centre w and the radius equal to twice the modulus of w.



Comment:
Generally well done, though some students were confused about the radius
Students were penalised if not enough information was presented

(iii) $|z - w| = |z|$ **1**

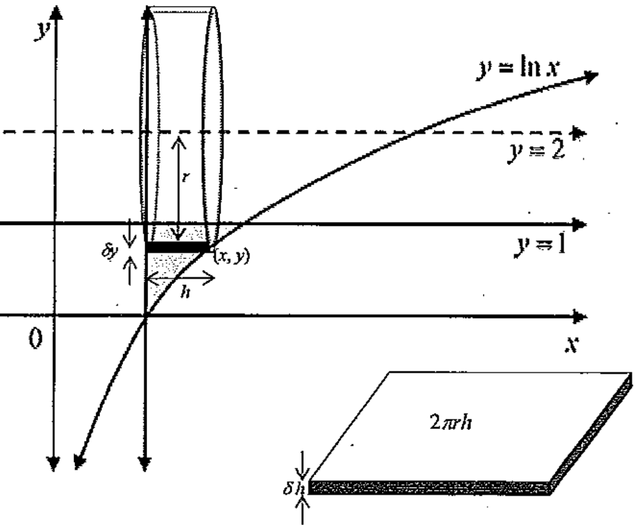
This is the perpendicular bisector of the line segment through w and the origin.



Comment:
This was not done well as students didn't read the question.
Sketch, on separate diagrams, the locus of P in each of the following cases making clear the relationship between the locus and the point Q.
It was not necessary to find the equation, but it was essential to indicate the perpendicular bisector.

Question 11 (continued)

(d) The region bounded by the curve $y = \ln x$, $x = 1$ and $y = 1$ is shaded in the diagram below. The region is rotated about the line $y = 2$ to form a solid. Using the method of cylindrical shells, find the volume of the solid formed. **4**



$h = x - 1$
$r = 2 - y$
Let δV be the volume of one shell of thickness δy.

$$\delta V \doteqdot 2\pi(2 - y)(x - 1)\,\delta y$$
$$= 2\pi(2 - y)(e^y - 1)\,\delta y$$
$$\therefore V = 2\pi\int_0^1 (2 - y)(e^y - 1)\,dy$$

$$V = 2\pi\int_0^1 \underbrace{(2 - y)}_{u}\underbrace{(e^y - 1)}_{dv}\,dy$$
$$= 2\pi\left[\underbrace{(2 - y)}_{u}\underbrace{(e^y - y)}_{v}\right]_0^1 - 2\pi\int_0^1 \underbrace{(-1)}_{du}\underbrace{(e^y - y)}_{v}\,dy$$
$$= 2\pi\left[(1)(e - 1) - (2)(1)\right] + 2\pi\int_0^1 (e^y - y)\,dy$$
$$= 2\pi(e - 3) + 2\pi\left[e^y - \frac{y^2}{2}\right]_0^1$$
$$= 2\pi(e - 3) + 2\pi\left[(e - \tfrac{1}{2}) - 1\right]$$
$$= 2\pi(e - 3 + e - \tfrac{3}{2})$$
$$= (4e - 9)\pi \text{ u}^3$$

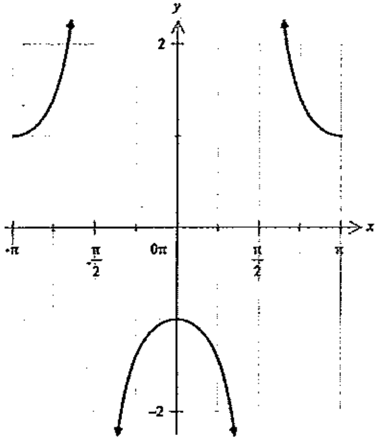
Comment:
Generally well done though no one did it the simplest way i.e. without expanding.

Of concern, was the handling of negative signs.

Solutions: SHS Maths Ext 2 Trial 2016

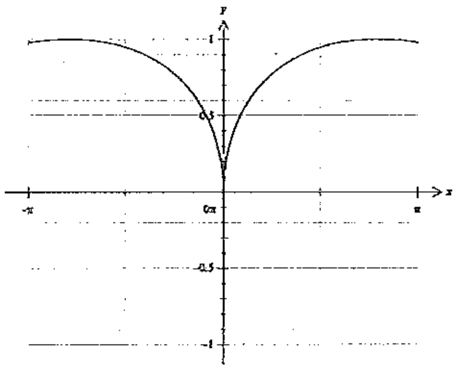
Question 12

(a) (i)
$$y=\frac{1}{\sin\left(x-\frac{\pi}{2}\right)}$$
$$=\frac{1}{-\sin\left(\frac{\pi}{2}-x\right)}$$
$$=\frac{-1}{\cos x}$$
$$=-\sec x$$



[Comments: This question was generally well understood. Many failed to limit their answer to the domain given, but were not penalized.]

(i) $y=\sin\sqrt{|x|}$



[Comments: Generally well understood, but not well drawn, in most cases. The function decreases away from the two values where it equals 1. Many lost 1 mark for these boundary values.]

(b) (i)
$$P(1)=1^5-1=0$$
$$P(\alpha)=\left(\text{cis}\frac{2\pi}{5}\right)^5-1=\text{cis}2\pi-1=0$$
$$P\left(\frac{1}{\alpha}\right)=\left(\text{cis}\frac{-2\pi}{5}\right)^5-1=\text{cis}(-2\pi)-1=0$$
$$P(\alpha^2)=\left(\text{cis}\frac{4\pi}{5}\right)^5-1=\text{cis}4\pi-1=0$$
$$P\left(\frac{1}{\alpha^2}\right)=\left(\text{cis}\frac{-4\pi}{5}\right)^5-1=\text{cis}(-4\pi)-1=0$$

Alternatively
$$P(\alpha)=\left(\text{cis}\frac{2\pi}{5}\right)^5-1=\text{cis}2\pi-1=0$$

Consider
$$P(\alpha^k)=(\alpha^k)^5-1=(\alpha^5)^k-1=1^k-1=0$$

Hence α^k is a root for all integral k, including 0, 2, -1, -2.

[Comments: Most used either of the two methods shown, mainly the first, but many solved the equation from scratch.]

(ii)
$$RHS=z^2(z-1)\left(\left(z+\frac{1}{z}\right)^2+\left(z+\frac{1}{z}\right)-1\right)$$
$$=z^2(z-1)\left(z^2+2+\frac{1}{z^2}+z+\frac{1}{z}-1\right)$$
$$=(z^3-z^2)\left(z^2+z+\frac{1}{z}+\frac{1}{z^2}+1\right)$$
$$=z^5+z^4+z+z^2+z^3-z^4-z^3-1-z-z^2$$
$$=z^5-1$$
$$=LHS$$

[Comments: This was very well done throughout.]

(iii)
$$\alpha+\frac{1}{\alpha}=\text{cis}\left(\frac{2\pi}{5}\right)+\text{cis}\left(\frac{-2\pi}{5}\right)$$
$$=\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{2\pi}{5}\right)-i\sin\left(\frac{2\pi}{5}\right)$$
$$=2\cos\left(\frac{2\pi}{5}\right)$$

Thus $\alpha^2(\alpha-1)\left(\left(2\cos\left(\frac{2\pi}{5}\right)\right)^2+2\cos\left(\frac{2\pi}{5}\right)-1\right)=0$

Now $\alpha^2\neq 0,\ \alpha-1\neq 0$

$$\therefore \left(2\cos\left(\frac{2\pi}{5}\right)\right)^2+2\cos\left(\frac{2\pi}{5}\right)-1=0$$

$$4\cos^2\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{2\pi}{5}\right)-1=0$$

[Comments: Whilst most managed to find the result, many failed to point out why the first two factors could be discounted, and thereby lost a mark.]

(iv) Thus $\cos\left(\frac{2\pi}{5}\right)=\frac{-2\pm\sqrt{4+16}}{8}$

$$=\frac{-2\pm\sqrt{20}}{8}$$

$$=\frac{\pm2\sqrt{5}-2}{8}$$

$$=\frac{\sqrt{5}-1}{4}\quad\text{(pos result, 1st quad)}$$

[Comments: Half a mark was lost by those who failed to explain why they chose the positive root.]

(c) Choose 2 vertices from 11, but remove 11 sides.

$${}^{11}C_2-11=44$$

Alternatively: Diagonals start from 11 vertices, and go to 8 from each, but this counts each diagonal twice. Hence

$$(11\times8)\div2=44$$

[Comments: Many candidates need to apply a reasonableness test to their answers, often in the billions.]

(d) (i) $y=x^{n-1}(1+x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx}=(n-1)x^{n-2}(1+x^2)^{\frac{1}{2}}+x^{n-1}\frac{2x}{2\sqrt{1+x^2}}$$

$$=(n-1)x^{n-2}\sqrt{1+x^2}+\frac{x^n}{\sqrt{1+x^2}}$$

[Comments: Generally very well answered.]

(ii) $I_n=\int\frac{x^n}{\sqrt{1+x^2}}dx$

$$=\int\left[\frac{d}{dx}\left(x^{n-1}(1+x^2)^{\frac{1}{2}}\right)-(n-1)x^{n-2}\sqrt{1+x^2}\right]dx$$

$$=x^{n-1}(1+x^2)^{\frac{1}{2}}-\int(n-1)x^{n-2}\sqrt{1+x^2}\,dx$$

$$I_n+(n-1)\int x^{n-2}\sqrt{1+x^2}\,dx=x^{n-1}(1+x^2)^{\frac{1}{2}}$$

$$I_n+(n-1)\int x^{n-2}\frac{(1+x^2)}{\sqrt{1+x^2}}dx=x^{n-1}(1+x^2)^{\frac{1}{2}}$$

$$I_n+(n-1)\int\left(\frac{x^{n-2}}{\sqrt{1+x^2}}+\frac{x^2}{\sqrt{1+x^2}}\right)dx=x^{n-1}(1+x^2)^{\frac{1}{2}}$$

$$I_n+(n-1)I_{n-2}+(n-1)I_n=x^{n-1}(1+x^2)^{\frac{1}{2}}$$

$$nI_n+(n-1)I_{n-2}=x^{n-1}(1+x^2)^{\frac{1}{2}}$$

$$\text{Hence } I_n+\frac{n-1}{n}I_{n-2}=\frac{x^{n-1}(1+x^2)^{\frac{1}{2}}}{n}$$

[Comments: Most attempted this part, and around half produced a satisfactory argument.]

(iii) $I_n+\frac{n-1}{n}I_{n-2}=\frac{x^{n-1}(1+x^2)^{\frac{1}{2}}}{n}$

$$I_3+\frac{3-1}{3}I_1=\frac{x^2(1+x^2)^{\frac{1}{2}}}{3}$$

$$I_3+\frac{2}{3}I_1=\frac{x^2\sqrt{1+x^2}}{3}$$

$$I_3=\frac{x^2\sqrt{1+x^2}}{3}-\frac{2}{3}I_1$$

Now $I_1=\int\frac{x}{\sqrt{1+x^2}}dx$

$$=\frac{1}{2}\int\frac{2x}{\sqrt{1+x^2}}dx$$

$$=\sqrt{1+x^2}+C$$

$$\therefore\int\frac{x^3}{\sqrt{1+x^2}}dx=\frac{x^2\sqrt{1+x^2}}{3}-\frac{2}{3}\sqrt{1+x^2}+C$$

[Comments: Generally well answered, by all those who attempted it.]

Q13 (a)(i) P(A, B) next to each other when n people are seated at table

Now no of ways without restriction $= (n-1)!$

From n people, no ways A, B together $= (n-2)!$ (treating A, B as one 'person')

$\therefore P(A, B \text{ together}) = \dfrac{2!(n-2)!}{(n-1)!} = p$

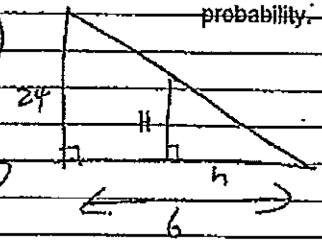
$= \dfrac{2}{n-1}$

(ii) $P(A, B, C \text{ together}) = \dfrac{(n-3)! \times 3!}{(n-1)!}$

$= \dfrac{6}{(n-1)(n-2)}$ ✓ 2

Students did this fairly well if they realised it was an arrangement around a circle. Most errors were because they didn't find the probability.

(b)

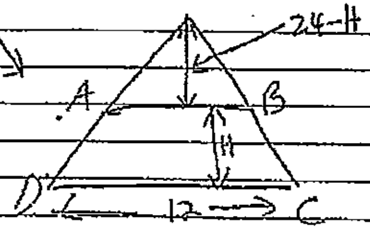


similar △'s

$\dfrac{h}{H} = \dfrac{6}{24}$

$\Rightarrow H = 4h$ ✓

2



$\dfrac{AB}{24-H} = \dfrac{12}{24} = \dfrac{1}{2}$

$AB = \dfrac{24-H}{2}$ ✓

$AB = \dfrac{24-4h}{2} = 12-2h$

First part was done well. The second part to find AB was done poorly. Some used the equation of a line successfully instead of similar triangles.

Q. 13 (b)

$V_{slice} = \frac{1}{2}(a+b)H \times \delta h$

$= \frac{1}{2}(AB+12)H\,\delta h$

$= \frac{1}{2}(12-2h+12)4h\,\delta h$ ✓

$V = \int_0^{12} 4h(12-h)\,dh$

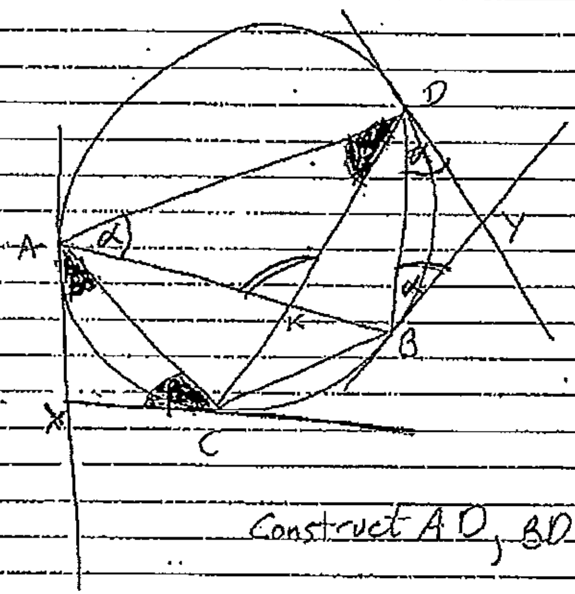
By symmetry of pyramid

$V = 2\int_0^{6} 4h(12-h)\,dh$ ✓

2

Students only received 2 marks if they had the volume as integral from 0 to 12 and then making it 2 times the integral from 0 to 6 with a comment on the symmetry of the pyramid.

13.(c)



Construct AD, BD, AC

Then let $\angle DBY = \alpha$ and $\angle ACX = \beta$

Then $\angle DAB = \alpha$ (Alt. segment Th)

$\angle CDB = \beta$ (same reason)

Then $\boxed{\angle AKD = 180 - (\alpha+\beta)}$ ①

3

Also $\angle CAX = \beta$ (Tangents from a point are equal ∴ Isosceles △)

and $\angle BDY = \alpha$ (same reason)

Then $\angle AXC = 180 - 2\beta$ and $\angle DYB = 180 - 2\alpha$

∴ $\angle AXC + \angle BYD = 360 - 2(\alpha+\beta)$

$= 2(180 - (\alpha+\beta)) = 2\angle AKD$ from ①

Students either saw the relationships required or didn't. Many assumed AXCK and DKYB were cyclic quadrilaterals without proof.

13.(d) $x^3 - x + 2 = 0$

$f'(x) = 3x^2 - 1$

For st pts $f'(x) = 0$

$\Rightarrow 3x^2 - 1 = 0$

$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

Sub into $f(x) = x^3 - x + 2$

For $x = \frac{1}{\sqrt{3}}$, $f(x) = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} + 2$

$= \frac{1 - 3 + 6\sqrt{3}}{3\sqrt{3}}$

$\neq 0$

For $x = \frac{-1}{\sqrt{3}}$, $f(x) = \frac{-1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} + 2$

$= \frac{-1 + 3 + 6\sqrt{3}}{3\sqrt{3}}$

$\neq 0$

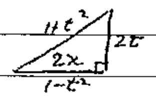
∴ Neither of zeros of $f'(x)$ are zeros of $f(x)$, ∴ $f(x)$ does not have a double root.

2

This question was done well by most students.

14) a) i) $2\sin 2x + \cos 2x = k$

let $t = \tan x$



$$2\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) = k$$

$$4t + 1 - t^2 = k + kt^2$$

$$(1+k)t^2 - 4t + (k-1) = 0$$

$$(1+k)\tan^2 x - 4\tan x + (k-1) = 0$$

ii) $$\tan x_1 + \tan x_2 = \frac{-(-4)}{1+k}$$

$$= \frac{4}{1+k}$$

$$\tan x_1 \tan x_2 = \frac{k-1}{1+k}$$

$$\tan(x_1 + x_2) = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2}$$

$$= \frac{\frac{4}{1+k}}{1 - \frac{k-1}{1+k}} \times \frac{1+k}{1+k}$$

$$= \frac{4}{1+k-(k-1)}$$

$$= \frac{4}{2}$$

$$= 2$$

COMMENT: I can't believe how many students tried to substitute k into the result. There is much more algebra to deal with and many were not successful.
Students had much more success with part(ii).

b) i) $(1+x+x^2)^n \equiv a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_{2n} x^{2n}$

let $x = 1$

$$(1+(1)+(1)^2)^n = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 + a_4(1)^4 + \ldots + a_{2n}(1)^{2n}$$

$$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + \ldots + a_{2n} \quad (1)$$

let $x = -1$

$$(1+(-1)+(-1)^2)^n = a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 + a_4(-1)^4 + \ldots + a_{2n}(-1)^{2n}$$

$$1 = a_0 - a_1 + a_2 - a_3 + a_4 - \ldots + a_{2n} \quad (2)$$

$(1) + (2)$

$$2(a_0 + a_2 + a_4 + \ldots + a_{2n}) = 3^n + 1$$

$$a_0 + a_2 + a_4 + \ldots + a_{2n} = \frac{3^n + 1}{2}$$

ii) a_r is the coefficient of x^r in the expansion of $(1+x+x^2)^n$.

$$(1+x+x^2)^n \equiv \left(x^2(1+x^{-1}+x^{-2})\right)^n$$

$$\equiv x^{2n}(1+x^{-1}+x^{-2})^n$$

a_r is the coefficient of x^{-r} in the expansion of $(1+x^{-1}+x^{-2})^n$

a_r is the coefficient of x^{2n-r} in the expansion of $x^{2n}(1+x^{-1}+x^{-2})^n$ ie $(1+x+x^2)^n$.

$\therefore a_{2n-r} = a_r$

COMMENT: This question was done particularly poorly. Students that tried to connect a_r with ${}^nC_r$ didn't get very far.

c) i) $\dfrac{z^2}{\bar{z}} = \dfrac{[r(\cos\theta + i\sin\theta)]^2}{[\overline{r(\cos\theta + i\sin\theta)}]}$

$= \dfrac{r^2(\cos 2\theta + i\sin 2\theta)}{r(\cos\theta - i\sin\theta)}$

$= r\dfrac{(\cos 2\theta + i\sin 2\theta)}{\cos(-\theta) + i\sin(-\theta)}$

$= r\left(\cos(2\theta - (-\theta)) + i\sin(2\theta - (-\theta))\right)$

$= r(\cos 3\theta + i\sin 3\theta)$

ii) $z^2 = i\bar{z}$

$\dfrac{z^2}{\bar{z}} = i$

$r(\cos 3\theta + i\sin 3\theta) = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$r = 1 \qquad 3\theta = \frac{\pi}{2} + 2k\pi$, where $k = 0, 1, -1$

$\theta = \dfrac{\pi}{6} + \dfrac{2k\pi}{3}$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, -\dfrac{\pi}{2}$

iii) $w = \cos\alpha + i(1 + \sin\alpha)$

$w = \cos\alpha + i + i\sin\alpha$

$w - i = \cos\alpha + i\sin\alpha$

$|w - i| = |\cos\alpha + i\sin\alpha|$

$= 1$

iv) $(w - i)^2 + 1 = i\bar{w}$

$(w - i)^2 = i\bar{w} - 1$

$= i\bar{w} + i^2$

$= i(\bar{w} + i)$

$= i(\overline{w - i})$

using (ii)

$w - i = 1(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}),\ 1(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}),\ 1(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}))$

$w - i = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i,\ -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i,\ -i$

$w = \dfrac{\sqrt{3}}{2} + \dfrac{3}{2}i,\ -\dfrac{\sqrt{3}}{2} + \dfrac{3}{2}i,\ 0$

COMMENT: Part (i) & (ii) were done particularly well by students.
Part (iv) was done poorly because students did not use part (ii) to solve the equation and those who did

eg $(w - i)^2 + 1 = i\bar{w}$

$(w - i)^2 + 1 = w^2$ (using (ii))

$\cancel{w^2} - 2wi \cancel{- 1} + \cancel{1} = \cancel{w^2}$

$-2wi = 0$

$w = 0$

had the problem that $|w| \neq 1$ which is needed

QUESTION 15 (X2)

(a) Prove that $f^{(n)}(x) = (-1)^n e^{-x}(x-n)$ (A)
where $f(x) = x e^{-x}$, for $n \in Z^+$.

Step I Show true for $n=1$

ie $f'(x) = (-1)^1 e^{-x}(x-1)$

now LHS $= x \times -e^{-x} + 1 \times e^{-x}$
$= -e^{-x}(x-1)$
$=$ RHS. $\therefore$ true.

Step II Assume the statement (A) is true for $n=k$.

ie. $f^{(k)}(x) = (-1)^k e^{-x}(x-k)$ [4]

Step III Use the assumption to show that (A) is true for $n=k+1$

ie. $f^{(k+1)}(x) = (-1)^{k+1} e^{-x}(x-(k+1))$

$$LHS = f'\left[(-1)^k e^{-x}(x-k)\right]$$
$$= (-1)^k e^{-x} \times 1 + -(-1)^k e^{-x}(x-k)$$
$$= (-1)^k e^{-x}\left[1-(x-k)\right]$$
$$= (-1)^{k+1} e^{-x}\left[-1+(x-k)\right]$$
$$= (-1)^{k+1} e^{-x}\left(x-(k+1)\right)$$
$$= RHS.$$

STEP IV By the Principle of Mathematical Induction the statement (A) is true for all $n \in Z^+$.

COMMENT The question was very well done with most students scoring [4 MARKS.]

(b) (i). $\ddot{x} = -k\dot{x}$

ie $\frac{dv_x}{dt} = -k v_x$

$$\frac{dt}{dv_x} = \frac{-1}{k v_x}$$

$$t = -\frac{1}{k}\ln v_x + c$$

when $t=0$, $v_x = U$

$$\therefore 0 = -\frac{1}{k}\ln U + c_1$$

$$c_1 = \frac{1}{k}\ln U.$$

$$\therefore t = -\frac{1}{k}\ln\frac{v_x}{U}$$ [3]

$$e^{-kt} = \frac{v_x}{U}$$

$$\boxed{v_x = U e^{-kt}}$$ (B)

$$\therefore \frac{dx}{dt} = U e^{-kt}$$ [OR. $\int_0^x dx = \int_0^t U e^{-kt}\,dt$]

$$x = -\frac{U}{k}e^{-kt} + c_2$$

now $x = 0$ when $t=0$

$$\therefore 0 = -\frac{U}{k}e^0 + c_2$$

$$c_2 = \frac{U}{k}$$

$$\therefore x = \frac{-U}{k}e^{-kt} + \frac{U}{k}$$

$$\boxed{x = \frac{U}{k}\left(1-e^{-kt}\right)} \text{ m.}$$

COMMENT The above was the most successful method. Some students started with $\ddot{x} = v\,\frac{dv}{dx}$ then struggled to get x into the answer.

(ii) We have that in vertical direction

$$\frac{dv}{dt} = g - kv$$

$$\therefore \frac{dt}{dv} = \frac{1}{g-kv}$$

$$t = \int \frac{dv}{g-kv}$$

$$= -\frac{1}{k}\ln(g-kv) + c_3$$

now $v=0$ when $t=0$.

$$\therefore 0 = -\frac{1}{k}\ln g + c_3$$

$$c_3 = \frac{1}{k}\ln g \qquad [3]$$

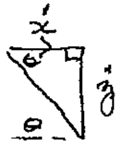
$$\therefore t = -\frac{1}{k}\ln\frac{g-kv}{g}$$

ie. $\frac{g-kv}{g} = e^{-kt}$

$$g - kv = g e^{-kt}$$

$$kv = g - g e^{-kt}$$

$$\boxed{v_y = \frac{g}{k}\left(1 - e^{-kt}\right)}$$



now. $\tan\theta = \frac{\dot{y}}{\dot{x}}$

$$= \frac{\frac{g}{k}\left(1-e^{-kt}\right)}{Ue^{-kt}} \leftarrow \text{from (i)} \quad (B)$$

$$= \frac{g}{kU}\left[e^{kt}-1\right]$$

$$\therefore \boxed{\theta = \tan^{-1}\left[\frac{g}{kU}\left(e^{kt}-1\right)\right]}$$

COMMENT This question proved difficult for many students. Some students tried to find $\frac{dy}{dx}$ and others incorrectly used $\tan\theta = \frac{y}{x}$ instead of $\frac{\dot{y}}{\dot{x}}$

(iii) $S^2 = v_x^2 + v_y^2$

$$= \frac{U^2}{e^{+2kt}} + \left[\frac{g}{k}\left(1 - \frac{1}{e^{kt}}\right)\right]^2 \quad (C) \qquad [2]$$

now $\frac{1}{e^{kt}} \to 0$ as $t \to \infty$.

$$\therefore S^2 \to U^2 \times 0 + \left[\frac{g}{k}(1-0)\right]^2$$

$$= 0 + \frac{g^2}{k^2} \quad \therefore \left[S = \sqrt{g^2/k^2} = g/k\right]$$

$\therefore \boxed{V = \frac{g}{k}}$ where V = terminal velocity.

COMMENT The question specifically asked "by considering the components of the velocity". (to justify it as a 2 MARK Question) most students failed to read this. and gave an answer based on $\dot{y} = 0$ which resulted in a loss of $\frac{1}{2}$ MARK OR sometimes 1 mark depending on mention of the horizontal case.

(IV) Let $S^2 = U^2 e^{-2kt} + V^2(1-e^{-kt})^2$ [from (C) in (III)]

ie $S^2 = U^2 e^{-2kt} + V^2(1 - 2e^{-kt} + e^{-2kt})$

$= (U^2+V^2)e^{-2kt} - 2V^2 e^{-kt} + V^2$ (D)

[This is a quadratic in the form aZ^2+bZ+c when a is (U^2+V^2) ie >0. with $Z = e^{-kt}$]

$= (U^2+V^2)\left[e^{-2kt} - \frac{2V^2}{U^2+V^2}e^{-kt}\right] + V^2$.

$= (U^2+V^2)\left[e^{-2kt} - \frac{2V^2}{U^2+V^2}e^{-kt} + \frac{(V^2)^2}{(U^2+V^2)^2}\right] + V^2 - \frac{(U^2+V^2)V^4}{(U^2+V^2)^2}$

$= (U^2+V^2)\left[e^{-kt} - \frac{V^2}{U^2+V^2}\right]^2 + V^2 - \frac{V^4}{U^2+V^2}$ [3]

$= (U^2+V^2)\left[e^{-kt} - \frac{V^2}{U^2+V^2}\right]^2 + \frac{V^2(U^2+V^2) - V^4}{U^2+V^2}$

$= (U^2+V^2)\left[e^{-kt} - \frac{V^2}{U^2+V^2}\right]^2 + \frac{U^2V^2 + V^4 - V^4}{U^2+V^2}$

$= (U^2+V^2)\left(e^{-kt} - \frac{V^2}{U^2+V^2}\right)^2 + \frac{U^2V^2}{U^2+V^2}$

$\therefore$ MIN of $\frac{U^2V^2}{U^2+V^2}$ when $e^{-kt} = \frac{V^2}{U^2+V^2}$

COMMENT: A very small number of students gained full marks. There were 3 different ways of doing this question. One using calculus and a little quadratic approach using $Z = \frac{-b}{2a}$.

Alternate [1] to (IV)

For (D) $(S^2)' = -2k(U^2+V^2)e^{-2kt} + 2V^2k\,e^{-kt}$ (E)

$(S^2)'' = 4k^2(U^2+V^2)e^{-2kt} - 2V^2k^2e^{-kt}$ (F)

Let $(S^2)' = 0$ in (E)

$\frac{2k(U^2+V^2)}{e^{2kt}} = \frac{2V^2k}{e^{kt}}$

$\therefore e^{kt} = \frac{U^2+V^2}{V^2}$

$\therefore$ in (F) $(S^2)'' = 4k^2(U^2+V^2) \times \frac{V^4}{(U^2+V^2)^2} - 2V^2k^2 \times \frac{V^2}{U^2+V^2}$

$= \frac{4k^2V^4}{U^2+V^2} - \frac{2k^2V^4}{U^2+V^2}$

$= \frac{2k^2V^4}{U^2+V^2}$

$> 0 \therefore$ MIN at $e^{kt} = \frac{U^2+V^2}{V^2}$

For $S^2 = (U^2+V^2)e^{-2kt} - 2V^2e^{-kt} + V^2$ (D) in (IV).

$\therefore$ sub. $e^{kt} = \frac{U^2+V^2}{V^2}$

$S^2_{MIN} = (U^2+V^2) \times \frac{V^4}{(U^2+V^2)^2} - 2V^2 \times \frac{V^2}{U^2+V^2} + V^2$

$= \frac{V^4}{U^2+V^2} - \frac{2V^4}{U^2+V^2} + V^2$

$= \frac{V^4 - 2V^4 + V^2(U^2+V^2)}{U^2+V^2}$

$= \frac{V^4 - 2V^4 + U^2V^2 + V^4}{U^2+V^2} = \frac{U^2V^2}{U^2+V^2} = W^2$

Alternate II in (iv)

We have the quadratic in e^{-kt}

ie. $S^2 = (U^2+V^2)e^{-2kt} - 2V^2 e^{-kt} + V^2$ (D)

$(U^2+V^2) > 0 \therefore$ MIN.

Axis is $\frac{V^2}{U^2+V^2}$ (ie. $az^2+bz+c=0$, $z=\frac{-b}{2a}$)

$\therefore$ sub back to (D)

$S^2_{MIN} = W^2 = (U^2+V^2)\times\frac{V^4}{(U^2+V^2)^2} - 2V^2\times\frac{V^2}{U^2+V^2} + V^2$

$= \frac{U^2V^2}{U^2+V^2}$ (as in previous solution)

Question 16 **SOLUTIONS**

(a)	(i)	Show that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$, by expanding the right hand side.	1

$$\begin{aligned} \text{RHS} &= \sin(A+B) - \sin(A-B) \\ &= \sin A\cos B + \sin B\cos A - (\sin A\cos B - \sin B\cos A) \\ &= 2\cos A\sin B \\ &= \text{LHS} \end{aligned}$$

(ii)	The distinct points $P(\theta, \sin\theta)$ and $Q(\phi, \sin\phi)$ lie on the curve $y = \sin x$, where x is measured in radians. Show the gradient of the chord PQ may be expressed as $$\frac{\sin\left(\frac{\phi-\theta}{2}\right)\cos\left(\frac{\phi+\theta}{2}\right)}{\left(\frac{\phi-\theta}{2}\right)}.$$ Deduce that if ϕ is approximately equal to θ then the gradient of gradient PQ is approximately equal to $\cos\theta$.	2

$$\begin{aligned} m_{PQ} &= \frac{\sin\underbrace{\theta}_{A+B} - \sin\underbrace{\phi}_{A-B}}{\theta-\phi} \\ &= \frac{2\cos A\sin B}{\theta-\phi} \\ &= \frac{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)}{\theta-\phi} \end{aligned}$$

$\left.\begin{aligned}\theta &= A+B \\ \phi &= A-B\end{aligned}\right\}+$	$\left.\begin{aligned}\theta &= A+B \\ \phi &= A-B\end{aligned}\right\}-$
$\theta+\phi = 2A$	$\theta-\phi = 2B$
$\therefore A = \frac{\theta+\phi}{2}$	$\therefore B = \frac{\theta-\phi}{2}$

For $\phi \doteqdot \theta$

$$\begin{aligned} \lim_{\phi\to\theta} m_{PQ} &= \lim_{\phi\to\theta}\frac{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)}{\theta-\phi} \\ &= 2\cos\theta\lim_{\phi\to\theta}\frac{\sin\left(\frac{\theta-\phi}{2}\right)}{\theta-\phi} \\ &= 2\cos\theta\times\frac{1}{2} \\ &= \cos\theta \end{aligned}$$

$\boxed{\lim_{u\to 0}\frac{\sin\frac{u}{2}}{u} = \frac{1}{2}}$

i.e. $m_{PQ} \doteqdot \cos\theta$

Comment:

Generally well done though in the Deduction part, students were penalised for not handling the limit properly.

Evidence of the rule $\sin x \doteqdot x$ for small x, or its equivalent, was necessary

(b) Ten people arrived in the Kingsmith Airport from London. It's late at night, only 4 immigration counters are open. In how many ways can 10 people line up in a 4-lane queue? **2**

Let the ten people be denoted by X, then we need 3 dividers to separate the Xs to 4 counters

e.g. XXX + XXX + XX + XX or XXXXX ++ XXX + XX or XXXXXXXXXX +++

There are $\dfrac{13!}{3!10!}$ to count this, BUT the people are distinct and there are 10! ways to arrange them.

$\therefore$ There are $\dfrac{13!}{3!10!}\times 10! = \dfrac{13!}{3!}$ ways for the people to line up

Alternatively with the 13 objects: XXXXXXXXXX +++ there are ${}^{13}C_3$ ways to choose where the "+" are put.

And so there are ${}^{13}C_3 \times 10! = {}^{13}P_{10}$ ways for the people to line up

NB This is essentially the "ring question" with 10 rings on 4 fingers.

Comment:

It's very hard to give credit to work that is badly set out or where the logic was not immediately evident. Students need to help the markers out and put more detail in.

(c) Let n be a positive integer.

(i) Show that $\dfrac{1}{1-t^2} = \left(1 + t^2 + t^4 + \ldots + t^{2n-2}\right) + \dfrac{t^{2n}}{1-t^2}$ for $t^2 \neq 1$. **2**

$$\begin{aligned}
\text{RHS} &= \left(\underbrace{1+t^2+t^4+\ldots+t^{2n-2}}_{a=1;\, r=t^2;\, n \text{ terms}}\right) + \frac{t^{2n}}{1-t^2} \\
&= \frac{1\left[1-(t^2)^n\right]}{1-t^2} + \frac{t^{2n}}{1-t^2} \\
&= \frac{1-t^{2n}+t^{2n}}{1-t^2} \\
&= \frac{1}{1-t^2} \\
&= \text{LHS}
\end{aligned}$$

Comment:

Various methods were used though students who used the GP method without indicating as such, or equivalent, were penalised.

Foolishly there were a few who tried induction. Some tried partial fractions, but forgot that the degree of the numerator must be smaller than the degree of the denominator.

(ii) For $-1 < x < 1$, show that $\displaystyle\int_0^x \frac{t}{1-t^2}\, dt = \ln\left(\frac{1}{\sqrt{1-x^2}}\right)$. **2**

$$\begin{aligned}
\text{LHS} &= -\frac{1}{2}\int_0^x \frac{-2t}{1-t^2}\, dt \\
&= -\frac{1}{2}\Big[\ln\left|1-t^2\right|\Big]_0^x \\
&= -\frac{1}{2}\Big[\ln\left|1-x^2\right| - \ln 1\Big]_0^x \\
&= -\frac{1}{2}\ln(1-x^2) \qquad [-1<x<1] \\
&= \frac{1}{2}\ln\sqrt{1-x^2} \\
&= \ln\left(\frac{1}{\sqrt{1-x^2}}\right) \\
&= \text{RHS}
\end{aligned}$$

Comment:

Generally well done, though some students need to recognise $\dfrac{f'(x)}{f(x)}$.

Question 16 (continued)

(iii) Using the above parts and by letting $x=\sqrt{\frac{8}{9}}$, deduce that **3**

$$\int_0^{\sqrt{\frac{8}{9}}} \frac{t^{2n+1}}{1-t^2}\,dt=\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k$$

From part (i) $\frac{1}{1-t^2}=\left(1+t^2+t^4+\ldots+t^{2n-2}\right)+\frac{t^{2n}}{1-t^2}$

$$\therefore \frac{t}{1-t^2}=\left(t+t^3+t^5+\ldots+t^{2n-1}\right)+\frac{t^{2n+1}}{1-t^2}$$

$$\therefore \frac{t^{2n+1}}{1-t^2}=\frac{t}{1-t^2}-\left(t+t^3+t^5+\ldots+t^{2n-1}\right)$$

$$=\frac{t}{1-t^2}-\sum_{k=1}^{n}t^{2k-1}$$

$$\int_0^x \frac{t^{2n+1}}{1-t^2}\,dt=\int_0^x\frac{t}{1-t^2}\,dt-\int_0^x\sum_{k=1}^{n}t^{2k-1}\,dt$$

$$=\ln\left(\frac{1}{\sqrt{1-x^2}}\right)-\sum_{k=1}^{n}\int_0^x t^{2k-1}\,dt$$

$$\left[\text{Let } x=\sqrt{\frac{8}{9}}\right]$$

$$=\ln\left(\frac{1}{\sqrt{1-\frac{8}{9}}}\right)-\sum_{k=1}^{n}\left[\frac{1}{2k}t^{2k}\right]_0^{\sqrt{\frac{8}{9}}}$$

$$=\ln\left(\frac{1}{\frac{1}{3}}\right)-\sum_{k=1}^{n}\frac{1}{2k}\left(\sqrt{\frac{8}{9}}\right)^{2k}$$

$$=\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k$$

Comment:
Generally well done, though there was a lot of fudging answers by students who didn't recognise to multiple by t.

Question 16 (continued)

(c) (iv) It can be shown that for $0\le t\le\sqrt{\frac{8}{9}}$, $\frac{t^{2n+1}}{1-t^2}\ge 0$ and $\frac{t^{2n+1}}{1-t^2}\le\frac{t^{2n+1}}{1-\frac{8}{9}}$. **3**

(Do NOT prove this)

Show that $0\le\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k\le\frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}$.

From (iii), $\int_0^x\frac{t^{2n+1}}{1-t^2}\,dt=\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k$

$\because 0\le t\le\sqrt{\frac{8}{9}}$, $\frac{t^{2n+1}}{1-t^2}\ge 0$, then $\int_0^x\frac{t^{2n+1}}{1-t^2}\,dt\ge 0$

$$\therefore 0\le\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k$$

$\because 0\le t\le\sqrt{\frac{8}{9}}$, $\frac{t^{2n+1}}{1-t^2}\le\frac{t^{2n+1}}{1-\frac{8}{9}}$ then $\int_0^{\sqrt{\frac{8}{9}}}\frac{t^{2n+1}}{1-t^2}\,dt\le\int_0^{\sqrt{\frac{8}{9}}}\frac{t^{2n+1}}{1-\frac{8}{9}}$

$$\therefore \int_0^{\sqrt{\frac{8}{9}}}\frac{t^{2n+1}}{1-t^2}\,dt\le 9\int_0^{\sqrt{\frac{8}{9}}}t^{2n+1}\,dt$$

$$=\left[\frac{1}{2n+2}t^{2(n+1)}\right]_0^{\sqrt{\frac{8}{9}}}$$

$$=\frac{1}{2n+2}\left(\frac{8}{9}\right)^{n+1}$$

$$\therefore 0\le\ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k\le\frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}$$

Comment:
Generally well done by those who had the time to get here!