Student Name: _____

St George Girls High School

Year 12 HSC Course

Mid-Course Examination

2016

Mathematics

General Instructions

- Reading time - 5 minutes
- Working time – 90 minutes
- Write using black or blue pen.
  Black pen is preferred.
- Board-approved calculators may be used.
- A reference sheet is provided at the back of this paper.
- Show relevant mathematical reasoning and/or calculations in Questions 6 – 10.

Section I	/5
Section II	
Q6	/12
Q7	/12
Q8	/12
Q9	/12
Q10	/12
Total Mark	/65
	%

Total Marks: 65

Section I
Total marks (5)
Attempt Questions 1 – 5
Use the multiple choice answer sheet provided.

Section II
Total marks (60)
Attempt Questions 6 – 10
Start each question in a new booklet.
All necessary working should be shown.
Marks may not be awarded for untidy or poorly set out work.

Section I

5 marks
Attempt Questions 1 – 5

Use the multiple-choice answer sheet provided for Questions 1 – 5.

1. What is the value of $\sum_{n=1}^{5}(4n-2)$?

   (A) 18
   (B) 39
   (C) 50
   (D) 90

2. For what values of x is the curve $f(x)=x^3+x^2$ concave down?

   (A) $x<-\frac{1}{3}$
   (B) $x>-\frac{1}{3}$
   (C) $x<-3$
   (D) $x>3$

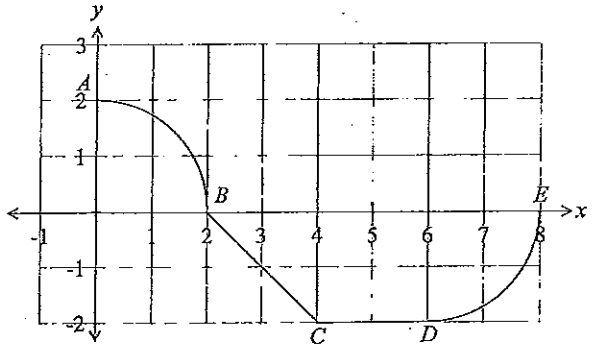
3. Evaluate $\int_0^5 dx$.

   (A) -5
   (B) 0
   (C) 5
   (D) x

4. The derivative of $(3x+2)\,e^{4x}$ is?

   (A) $11e^{4x}-12xe^{4x}$
   (B) $11e^{4x}+12xe^{4x}$
   (C) $12e^{4x}+11e^{4x}$
   (D) $12e^{4x}-11e^{4x}$

5. The graph of the function $y = f(x)$ consists of a quarter of a circle AB, a straight-line segment BC, a horizontal straight-line segment CD, and a quarter circle DE.



For what values of x is the function increasing?

(A) $0 < x < 2$

(B) $2 < x < 4$

(C) $4 < x < 6$

(D) $6 < x < 8$

Section II

60 marks
Attempt Questions 6 – 10

Start each question in a new writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (12 marks) Start this question in a new writing booklet. **Marks**

a) Find (i) $\int (3x+5)^2 dx$. **2**

(ii) $\int \frac{x^5-1}{x^2}\, dx$. **2**

b) The third term of a geometric series is $\frac{3}{4}$ and the seventh term is 12.

(i) Find the common ratio. **2**

(ii) Find the first term. **1**

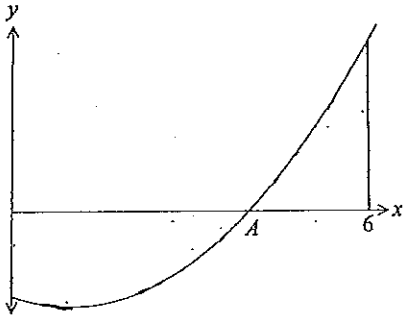
(iii) What is the tenth term? **1**

c) Use the trapezoidal rule with three function values to approximate the area bounded by the curve $y = \frac{2}{3}\sqrt{9-x^2}$, the coordinate axes and the line $x = 2$. **2**
Give your answer correct to two decimal places.

d) The graph $y = f(x)$ passes through the point $(1, 4)$ and $f'(x) = 3x^2 - 2$. **2**
Find the expression for $f(x)$.

Question 7 (12 marks) Start this question in a new writing booklet. **Marks**

a) The diagram below shows the graph of $y = x^2 - 2x - 8$ for $x \geq 0$.



Not to scale

(i) What are the coordinates of A? 1

(ii) Find the area bounded by the x-axis, the curve $y = x^2 - 2x - 8$ and the lines $x = 0$ and $x = 6$. 2

b) A function $f(x)$ is defined by $f(x) = 2x^3 - 3x^2$.

(i) Find all the solutions for $f(x) = 0$. 1

(ii) Find the turning points for the curve $y = f(x)$ and determine their nature. 3

(iii) Find the coordinates of the point of inflexion. 2

(iv) Sketch the graph of $y = f(x)$ showing the essential features. 2

(v) Find the values of x for which $f(x) < 0$. 1

Question 8 (12 marks) Start this question in a new writing booklet. **Marks**

a) Find the equation of the tangent to the curve $y = x^2 e^{3x}$ at the point where $x = 1$. 3

b) The region bounded by the curve $y = 2 - \sqrt{x}$ and the y-axis between $y = 0$ and $y = 2$ is rotated about the y-axis to form a solid. Find the volume of the solid in simplest exact form. 3

c) Dimitri invests $\$P$ at 8% per annum compounded annually. He plans to withdraw \$5000 at the end of each year for six years to cover university fees.

(i) Write down an expression for the amount $\$A_1$ remaining in the account following the withdrawal of the first \$5000. 1

(ii) Find an expression for the amount $\$A_3$ remaining in the account after the third withdrawal. 2

(iii) How much does Dimitri need to invest if the account balance is to be \$0 at the end of the six years? 3

Question 9 (12 marks) Start this question in a new writing booklet. **Marks**

a) Maria starts on a salary of \$55 000 with an annual increase of \$1650. What is the total amount Maria would earn in twelve years of employment? **2**

b) The table below shows the values of the function $y=10^x$ for five values of x. **2**

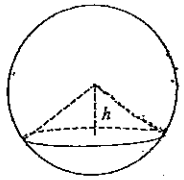
x	0	0.25	0.5	0.75	1
10^x	1	1.7783	3.1623	5.6234	10

Use Simpson's rule and the information in the table to estimate $\int_0^1 10^x dx$.

Give your answer correct to 4 significant figures.

c) Elisabeth deposits \$25 000 into an account at the beginning of each year for ten years. If the account earns interest at 6% p.a. compounded yearly, find the amount of money in the account at the end of ten years. **3**

d) The diagram shows a cone in a sphere. The vertex of the cone is at the centre of the sphere. The radius of the sphere is 12 cm.



(i) Show that the volume of the cone is $V=\frac{\pi}{3}(144h-h^3)$. **2**

(ii) Show that the maximum volume of the cone is $\frac{\sqrt{3}}{18}$ of the volume of the sphere. **3**

Question 10 (12 marks) Start this question in a new writing booklet. **Marks**

a) Solve $3^{2x+1}=\frac{1}{27}$. **2**

b) Express $0.5\dot{7}$ using the sum of an infinite series, and hence express $0.5\dot{7}$ in simplest rational form. **2**

c) Algebraically, determine the number of points of intersection of the curves $y=2e^{-x}+1$ and $y=2-\frac{1}{e^{2x}}$. **3**

d) Consider $y=e^{kx}$ where k is a constant.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **2**

(ii) Determine the value of k for which $y=e^{kx}$ satisfies the equation $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y=0$ **3**

End of Examination

Student Name: _____

Class Teacher: _____

Section I

5

Year 12 Mid-HSC Course

Mathematics

Multiple-choice Answer Sheet - Questions 1 – 5

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2+4=$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct ↙

A ⊗ B ⊗ C ○ D ○

1. A ○ B ○ C ● D ○ ✓
2. A ● B ○ C ○ D ○ ✓
3. A ○ B ⊗ C ● D ○ 3. (C) ✓
4. A ○ B ● C ○ D ○ ✓
5. A ○ B ○ C ○ D ● ✓

QUESTION NUMBER: 6 (12)

NAME: _____

CLASS: _____

TEACHER: _____

a. $\int (3x+5)^2\,dx$

$= \frac{(3x+5)^3}{3\times 3} + C$ ✓

$= \frac{(3x+5)^3}{9} + C$ ✓

ii. $\int \frac{x^5-1}{x^2}\,dx$

$= \int x^3 - \frac{1}{x^2}\,dx$

$= \int x^3 - x^{-2}\,dx$ ✓

$= \frac{x^4}{4} - \frac{x^{-1}}{-1} + C$

$= \frac{1}{4}x^4 + \frac{1}{x} + C$ ✓

b. $T_3 = 3/4 \quad ar^{n-1} : ar^2$

$T_7 = 12 \quad ar^{n-1} = ar^6$

$ar^2 = 3/4 \quad ar^6 = 12$ (2)

$a = \frac{3}{4r^2}$ (1)

sub (1) → (2)

$\left(\frac{3}{4r^2}\right) r^6 = 12$

$\frac{3}{4} r^4 = 12$

$r^4 = 16$ ✓

$r = \pm 2$ ✓

$r = \pm 2$

ii) $ar^2 = 3/4$

$a \times 4 = 3/4$

$a = 3/16$ ✓

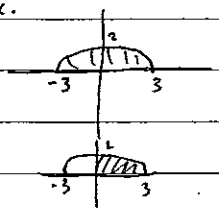
iii) $T_{10} = a \times r^{n-1}$

$= 3/16 \times 2^9$ or $3/16 \times -2^9$

$= 96$ or -96

$\therefore T_{10} = 96$ or $T_{10} = -96$ ✓

c.



x	0	1	2
y	2	$\frac{2\sqrt{8}}{3}$	$\frac{2\sqrt{5}}{3}$

$\frac{b-a}{n} = \frac{2-0}{2}$

$= 1$

$A \approx \frac{1}{2}\left[2 + \frac{2\sqrt{5}}{3} + 2\left(\frac{2\sqrt{8}}{3}\right)\right]$ ✓

$= \frac{1}{2} \times 7.261948151$

$= 3.630974076$ ✓

$= 3.63\ u^2$ (2d.p.)

d. $f'(x) = 3x^2 - 2$

$f(x) = \frac{3x^2}{3} - 2x + C$

$y = x^3 - 2x + C$

$4 = 1^3 - 2(1) + C$ ✓

$4 = 1 - 2 + C$

$4 = -1 + C$

$5 = C$ ✓

$\therefore y = x^3 - 2x + 3$ ✗

$\therefore y = x^3 - 2x + 5$

QUESTION NUMBER: 7

11

NAME:

CLASS:

TEACHER:

a) $y = x^2 - 2x - 8$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4, \quad x = -2$

but A > 0

$\therefore x = 4$ ✓

$\therefore A(4, 0)$

$A = \left| \int_0^4 x^2 - 2x - 8\ dx \right| + \int_4^6 x^2 - 2x - 8\ dx$

$= \left| \left[\frac{x^3}{3} - \frac{2x^2}{2} - 8x\right]_0^4 \right| + \left[\frac{x^3}{3} - \frac{2x^2}{2} - 8x\right]_4^6$

$= \left| \left[\frac{1}{3}x^3 - x^2 - 8x\right]_0^4 \right| + \left[\frac{1}{3}x^3 - x^2 - 8x\right]_4^6$

$= \left| \left[\frac{64}{3} - 16 - 32 - (0-0-0)\right] \right| + \left(72 - 36 - 48 - \left(\frac{64}{3} - 16 - 32\right)\right)$

$= \frac{80}{3} + \left(72 - 36 - 48 + \frac{80}{3}\right)$

$= \frac{80}{3} + \frac{44}{3}$

$= \frac{124}{3}$

$= 41.\dot{3}\ u^2$ ✓ 2

b. $f(x) = 2x^3 - 3x^2$

$2x^3 - 3x^2 = 0$

$x^2(2x - 3) = 0$

$x = 0, \quad x = \frac{3}{2}$ ✓ 1

If $y = 2x^3 - 3x^2$

$y' = 6x^2 - 6x$

$6x^2 - 6x = 0$

$6x(x - 1) = 0$

$x = 0, \quad x = 1$

when $x = 0$,

$y = 2(0)^3 - 3(0)^2$

$= 0$

$(0, 0)$

when $x = 1$,

$y = 2(1)^3 - 3(1)^2$

$= 2 - 3$

$= -1$

$(1, -1)$

ii. continued...

$y' = 6x^2 - 6x$

$y'' = 12x - 6$

at $x = 0$,

$y'' = 0 - 6$

$= -6$

< 0

$\therefore$ concave down,

max turning point at $(0, 0)$

at $x = 1$

$y'' = 12 - 6$

$= 6$

> 0 ✓

$\therefore$ concave up.

minimum turning point at $(1, -1)$

3

iii. $y'' = 0$

$\therefore 12x - 6 = 0$

$12x = 6$

$x = 1/2$

when $x = 1/2$

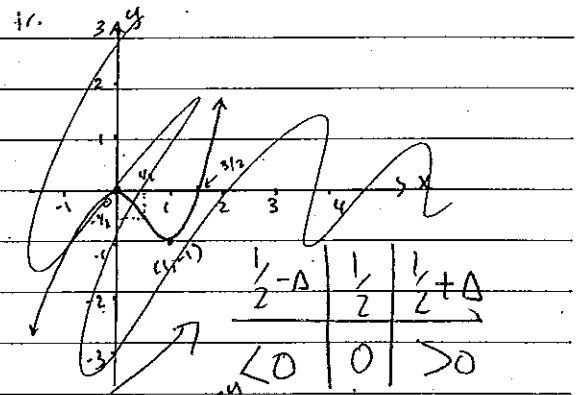
$y = 2(\frac{1}{2})^3 - 3(\frac{1}{2})^2$

$= \frac{2}{8} - \frac{3}{4}$

$= -1/2$

$= (1/2, -1/2)$

iv.



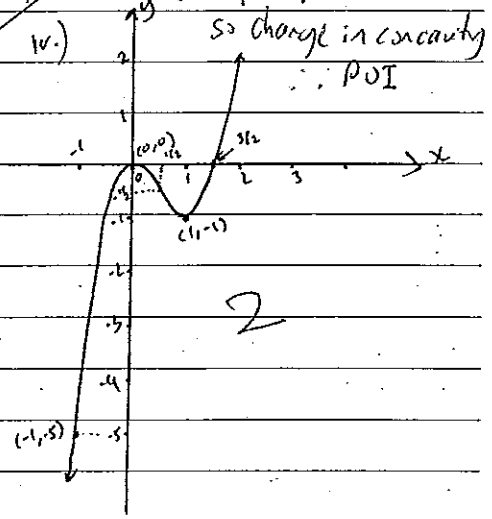
$\frac{1}{2} - \Delta$	$\frac{1}{2}$	$\frac{1}{2} + \Delta$
< 0	0	> 0

Check

So change in concavity

$\therefore$ POI

iv.)



v. $f(x) < 0$

when $0 < x < 3/2$

and $x < 0$

2

QUESTION NUMBER: 8 Part 1

NAME:

CLASS:

TEACHER:

12

a. $y = x^2 e^{3x}$

$u = x^2$

$u' = 2x$

$v = e^{3x}$

$v' = 3e^{3x}$

$y' = 2xe^{3x} + 3e^{3x}x^2$

$= xe^{3x}(2 + 3x)$

at $x = 1$,

$y' = 1 \times e^3 (2 + 3)$

$= e^3 \times 5$

$= 5e^3$

when $x = 1$,

$y = (1)^2 e^3$

$= e^3$

$(1, e^3)$

equation of tangent:

$y - e^3 = 5e^3(x - 1)$

$y - e^3 = 5e^3 x - 5e^3$

$y = 5e^3 x - 4e^3$

3

b. $y = 2 - \sqrt{x}$

$\sqrt{x} = 2 - y$

$x = (2 - y)^2$

$x^2 = (2 - y)^4$

$V = \pi \int_0^2 x^2 \, dy$

$= \pi \int_0^2 (2 - y)^4 \, dy$

$= \pi \left[\frac{(2-y)^5}{5} \right]_0^2$

$= \pi \left(0 - \left(-\frac{32}{5} \right) \right)$

$= \pi \times 32/5$

$= \frac{32\pi}{5} u^3$

3

c. i) $A_n = P(1+\frac{r}{100})^n$ refers to

$A_1 = P(1.08) - 5000$

ii) $A_2 = A_1(1+\frac{r}{100})^2 - 5000$

$= (P(1.08) - 5000)\,1.08^2 - 5000$

$= P(1.08)^3 - 5000(1.08^2) - 5000$

$A_2 = P(1.08)^3 - 5000(1.08^2 + 1)$

$A_3 = A_2(1.08)^2 - 5000$

$= (P(1.08)^3 - 5000(1.08^2+1))(1.08)^2 - 5000$

$= P(1.08)^5 - 5000(1.08)^4 - 5000(1.08)^2 - 5000$

$= P(1.08)^5 - 5000(1.08^4 + 1.08^2 + 1)$

iii. $A_6 = P(1.08)^{11} - 5000(1.08^{10} + \ldots + 1) = 0$ ($a=1$, $r=1.08^2$, $n=6$)

$= P(1.08)^{11} - 5000\left(\frac{1((1.08^2)^6 - 1)}{1.08^2 - 1}\right) = 0$

$P(1.08)^{11} = \$45618.09245$

$P = \$19564.81793$

$= \$19564.82$

QUESTION NUMBER: 8 Part 2 NAME: 1

CLASS:

TEACHER:

c. continued.

ii) $A_2 = A_1(1+\frac{r}{100}) - 5000$

$= (P(1.08) - 5000)1.08 - 5000$

$A_2 = P(1.08)^2 - 5000(1.08+1)$

$A_3 = A_2(1.08) - 5000$

$= (P(1.08)^2 - 5000(1.08+1))1.08 - 5000$

$= P(1.08)^3 - 5000 \times 1.08^2 - 5000 \times 1.08 - 5000$

$= P(1.08)^3 - 5000(1.08^2 + 1.08 + 1)$ 2

iii) $A_6 = P(1.08)^6 - 5000(1.08^5 + 1.08^4 + \ldots + 1)$

$a=1$, $r=1.08$, $n=6$

$A_6 = 0$

$\therefore P(1.08)^6 - 5000\left(\frac{1(1.08^6 - 1)}{1.08 - 1}\right)$

$= P(1.08)^6 - 36679.64518$

$P = \frac{36679.64518}{(1.08)^6}$

$= \$23114.39832$

$= \$23114.40$ 3

QUESTION NUMBER: 9 (11)

NAME:

CLASS:

TEACHER:

a) 55000 + 56650 + ...

$S_n = \frac{n}{2}(2a + (n-1)d)$

$a = 55000 \quad d = 1650$

$S_{12} = \frac{12}{2}(110000 + (11)1650)$ ✓

$= 6(128150)$

$= \$768900$ ✓ (2)

b. $y = 10^x$

$\frac{b-a}{n} = \frac{1-0}{4}$

$= 1/4$

$A \approx \frac{1}{12}(1 + 10 + 2(3.1623) + 4(1.7783 + 5.6234))$

$\approx \frac{1}{12}(11 + 6.3246 + 29.6068)$

$\approx \frac{1}{12} \times 46.9314$ ✓

$= 3.91095$ (2)

$= 3.911 \text{ u}^2$ ✓ (4 sig figs)

c. ~~P~~ 25000

$n = 10$

$A_{10} = 25000(1.06)^{10}$

$A_9 = 25000(1.06)^9$

$\vdots \quad \downarrow$

$A_1 = 25000(1.06)$

$S_{10} = 25000(1.06 + 1.06^2 + \cdots + 1.06^{10})$ ✓

$a = 1.06, n = 10, r = 1.06$

$S_{10} = 25000 \times \frac{1.06(1.06^{10} - 1)}{1.06 - 1}$ ✓ (3)

$= 25000 \times 13.97164264$

$= \$349291.066$

-9-

-01-

d. V of cone: ~~$\frac{8}{3}\pi r^3$~~

$= \frac{1}{3} \times \pi r^2 h$

$= \frac{1}{3} \times \pi \times (\sqrt{144 - h^2})^2 \times h$

$= \frac{\pi}{3} \times (144 - h^2) \times h$

$= \frac{\pi}{3} \times (144h - h^3)$

$V = \frac{\pi}{3}(144h - h^3)$ ✓ (2)

h, 12, $r = \sqrt{12^2 - h^2}$, $r = \sqrt{144 - h^2}$

$V = \frac{\pi}{3}(144h - h^3)$

$= \frac{144\pi}{3}h - \frac{\pi}{3}h^3$

$\frac{dV}{dh} = \frac{144\pi}{3} - \pi h^2$

$\frac{144\pi}{3} - \pi h^2 = 0$

$\pi h^2 = \frac{144\pi}{3}$

$h^2 = \frac{144}{3}$ (1)

$h = \frac{12}{\sqrt{3}}$ ✓ as $(h > 0)$

$= 4\sqrt{3}$

$\frac{144}{3} = 48$!

$\frac{d^2V}{dh^2} = -2\pi h$

at $h = 4\sqrt{3}$, $\frac{d^2V}{dh^2}$ = -ve. So maximum

$\therefore$ if $h = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

max. $V_{cone} = \frac{\pi}{3}(576\sqrt{3} - 192\sqrt{3})$

$= \frac{\pi}{3}(384\sqrt{3})$

$= 128\sqrt{3}\pi \text{ u}^3$

Show max. when $h = 4\sqrt{3}$ (-1)

$V_{sphere} = \frac{4}{3}\pi r^3$ 11 $V = ?$

$= 4/3 \times \pi \times 12^3$

$= 2304\pi \text{ u}^3$

$V_{cone} \div \frac{\sqrt{3}}{18}$

$= 128\sqrt{3}\pi \div \frac{\sqrt{3}}{18}$

$= 2304\pi$

$\therefore$ max. volume of cone is $\frac{\sqrt{3}}{18}$ the volume of the sphere ✓ (1)

QUESTION NUMBER: 10

10/12

NAME:

CLASS:

TEACHER:

a. $\ln 3^{2x+1} = \ln \frac{1}{27}$

$(2x+1)\ln 3 = \ln \frac{1}{27}$

$2x+1 = \frac{\ln 1/27}{\ln 3}$

$2x+1 = -3$

$2x = -4$

$x = -2$ ✓ 2

b. let $x = 0.5\dot{7}$ → $0.5 + [0.07 + 0.007 + 0.0007 + \ldots]$ GP.

$10x = 5.\dot{7}$

$10x - x = 5.\dot{7} - 0.5\dot{7}$

$9x = 5.2$

$x = \frac{5.2}{9}$

$= \frac{26}{45}$

need to use sum of infinite series.

$a = 0.07$

$r = \frac{1}{10}$

$0.5 + \left(\frac{a}{1-r}\right)$

$0.5 + \left(\frac{7/100}{9/10}\right)$

$= \frac{1}{2} + \frac{7}{90} = \frac{26}{45}$

c. $2e^{-x} + 1 = 2 - e^{-2x}$

$e^{-2x} + 2e^{-x} - 1 = 0$

let $e^{-x} = u$

$\therefore u^2 + 2u - 1 = 0$ ✓

$u = \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2}$ 3

$= \frac{-2 \pm \sqrt{8}}{2}$

$= \frac{-2 \pm 2\sqrt{2}}{2}$

$= -1 \pm \sqrt{2}$

$u = -1 \pm \sqrt{2}$ ✓

$e^{-x} = -1 + \sqrt{2}$

$\ln e^{-x} = \ln(-1+\sqrt{2})$

$-x = \ln(-1+\sqrt{2})$

$e^{-x} = -1 - \sqrt{2}$

$\ln e^{-x} = \ln(-1-\sqrt{2})$

∴ no solution

∴ only 1 point of intersection.

d. $y = e^{kx}$

(i) $\frac{dy}{dx} = ke^{kx}$ ✓

$\frac{d^2y}{dx^2} = k^2e^{kx}$ ✓ 2

(ii) $k^2e^{kx} + 7ke^{kx} + 12e^{kx} = 0$

$e^{kx}(k^2 + 7k + 12) = 0$ ✓

$e^{kx}(k+3)(k+4) = 0$ 3

$e^{kx} = 0$

∴ no solution. ✓

$k+3 = 0$

$k = -3$ ✓

and.

$k+4 = 0$

$k = -4$ ✓