Sydney Girls High School

2016

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- The Mathematics Reference Sheet is provided.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 3 – 6

10 Marks
- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 7 – 15

90 Marks
- Attempt Questions 11 – 16
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 2 hour and 45 minutes for this section

Name:	**THIS IS A TRIAL PAPER ONLY**
Teacher:	It does not necessarily reflect the format or the content of the 2016 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

1. The equation $24x^3-12x^2-6x+1$ has roots α, β and γ.
   What is the value of α if $\alpha=\beta+\gamma$?

   (A) $-\frac{1}{2}$
   (B) $\frac{1}{4}$
   (C) $\frac{1}{2}$
   (D) 1

2. Find $\int \frac{dx}{x^2-4x+13}$.

   (A) $\frac{1}{9}\tan^{-1}\left(\frac{x-2}{9}\right)+C$
   (B) $\frac{1}{9}\tan^{-1}\left(\frac{x-2}{3}\right)+C$
   (C) $\frac{1}{3}\tan^{-1}\left(\frac{x-2}{9}\right)+C$
   (D) $\frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$

3. Find the values of $\arg(z)$, given that z represents the solutions to the equation $z^2=1+i$.

   (A) $\frac{\pi}{8}$ and $\frac{15\pi}{8}$
   (B) $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$
   (C) $-\frac{\pi}{8}$ and $\frac{7\pi}{8}$
   (D) $-\frac{7\pi}{8}$ and $\frac{\pi}{8}$

4. A disc of radius 4 cm is spinning such that a point on the circumference is moving with a speed of 80 cm / min.

What is the angular speed of the disc (in revolutions per minute) ?

(A) 320 rpm

(B) 20 rpm

(C) $\frac{10}{\pi}$ rpm

(D) $\frac{160}{\pi}$ rpm

5. The hyperbola $16x^2 - 9y^2 = 144$ has foci at $S(5, 0)$ and $S'(-5, 0)$.

What are the equations of the directrices?

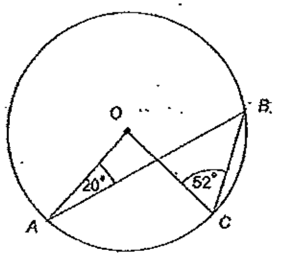
(A) $y = \frac{9}{5}$ and $y = -\frac{9}{5}$

(B) $x = \frac{9}{5}$ and $x = -\frac{9}{5}$

(C) $y = \frac{12}{5}$ and $y = -\frac{12}{5}$

(D) $x = \frac{12}{5}$ and $x = -\frac{12}{5}$

6. In the diagram below, O is the centre of the circle.



Given $\angle OAB = 20°$ and $\angle OCB = 52°$, what is the size of $\angle ABC$?

(A) 32

(B) 49

(C) 56

(D) 64

7. The horizontal base of a solid is the circle $x^2 + y^2 = 1$. Each cross section taken perpendicular to the x axis is a right-angled isosceles triangle with one of its shorter sides in the base of the solid. Which of the following is an expression for the volume of the solid ?

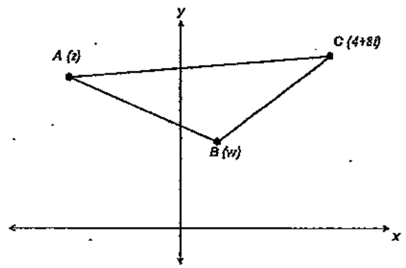
(A) $\frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx$

(B) $\int_{-1}^{1}(1-x^2)\,dx$

(C) $\frac{3}{2}\int_{-1}^{1}(1-x^2)\,dx$

(D) $2\int_{-1}^{1}(1-x^2)\,dx$

8. The Argand diagram below shows the triangle ABC where A represents the complex number $z = a + ib$ (where a and b are real), B represents the complex number ω and C represents $4+8i$.



Given triangle ABC is isosceles and $\angle ABC = 90°$, which of the following represents ω ?

(A) $\dfrac{(a-b-4)+i(a+b+4)}{2}$

(B) $\dfrac{(a+b+4)-i(a+b+4)}{2}$

(C) $\dfrac{(a+b+12)+i(a+b+4)}{2}$

(D) $\dfrac{(a-b+12)+i(a+b+4)}{2}$

9. Tom and Joanne each choose a **different** number at random from the integers 1, 2, 3,……., 20.

What is the probability that the sum of the numbers is 20?

(A) $\frac{1}{2}$

(B) $\frac{1}{20}$

(C) $\frac{1}{19}$

(D) $\frac{9}{190}$

10. Given $\frac{1}{r(r+1)(r+2)}=\frac{1}{2}\left[\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$, find

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)}.$$

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) $\frac{2}{9}$

Section II

Total 60 marks
Attempt Questions 11–16
Allow about 2 hour and 45 minutes for this section

Answer all questions, starting each question on a new page.
In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

Marks

Start each question on a NEW sheet of paper.

(a) Consider the complex numbers $z = 2 + 5i$ and $w = 3i$.

(i) Express $z + w$ in the form $a + ib$, where a and b are real. 1

(ii) Evaluate $|zw|$. 1

(iii) Find the value of $\arg(w^{71})$. 1

(b) Let α, β and γ be the roots of the equation 3

$$(3+i)z^3 + (2i-19)z^2 + (5+2i)z - 3i = 0.$$

Find the value of $\alpha+\beta+\gamma$, expressing your answer in the form $a + ib$ where a and b are real.

(c) Find :

(i) $\int \frac{x}{x^2+6x+25}dx$ 3

(ii) $\int \frac{e^{-2x}}{e^{-x}+5}dx$ 3

(d) (i) Expand and simplify $\sin(A+B)+\sin(A-B)$. 1

(ii) Hence, find $\int \sin 10x\cos 7x dx$. 2

End of Question 11

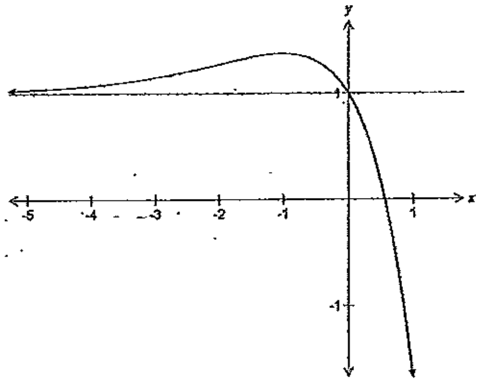
Question 12 (15 marks)

Marks

Start each question on a NEW sheet of paper.

(a) Sketch the region on the Argand diagram representing the intersection of $|z| < 5$ and $|z + \bar{z}| \geq 2$. **3**

(b) The graph of the function $y = f(x)$ is shown below.



Sketch the following curves, showing any key features.

(i) $y = [f(x)]^2$ **2**

(ii) $y = \ln[f(x)]$ **2**

(iii) $|y| = \frac{1}{f(x)}$ **3**

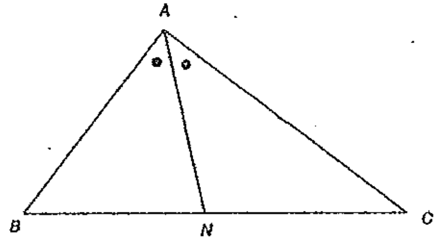
(c) One end of a light inextensible string of length 3 metres is fastened to a fixed point O on a smooth horizontal table. Masses of 3 kg and 5 kg are attached to the string at A and B respectively such that $OA: OB = 3:1$. Note that A is at the end of the string. Given that the system rotates about O at 2π radians per second, find in simplest form the ratio of the tensions in the string, $T_{OB} : T_{AB}$. **3**

Question 12 continues on the next page.

Question 12 (continued)

Marks

(d) In the diagram below, AN bisects $\angle ABC$. Using trigonometry, prove that $\frac{AB}{BN} = \frac{AC}{CN}$. **2**



End of Question 12

Question 13 (15 marks)

Marks

Start each question on a NEW sheet of paper.

(a) Find the acute angle between the tangent to the curve $x^2 + xy + 2y^2 = 28$ at the point $(-2, -3)$, and the line $y = x$. Give your answer to the nearest degree. 4

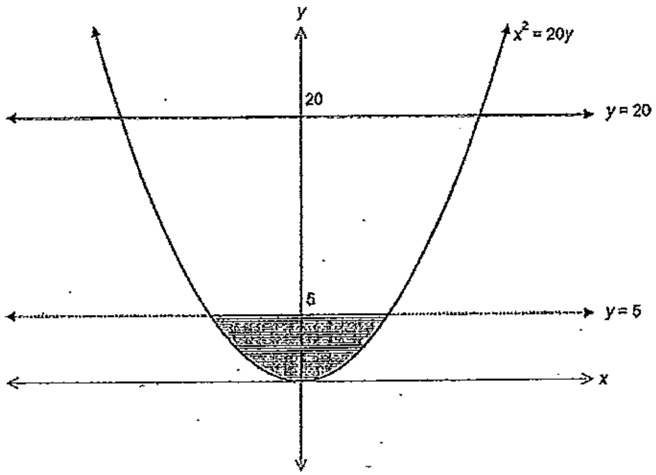
(b) Let α, β and γ be the roots of the equation

$$x^3 - 5x^2 + 7x - 18 = 0$$

(i) Find the cubic equation that has roots $4 + \alpha^2, 4 + \beta^2$ and $4 + \gamma^2$. 3

(ii) Hence, find the value of $\alpha^2 + \beta^2 + \gamma^2$. 1

(c) The area bounded by the parabola $x^2 = 20y$ and the line $y = 5$ (as shown below) is rotated about the line $y = 20$. Find the volume of the solid formed. 4



(d) Consider the polynomial $P(x) = 8x^4 + 28x^3 + 18x^2 - 27x - 27$.

(i) Find all zeros of $P(x)$, given that it has a zero of multiplicity 3. 2

(ii) Hence, sketch $y = P(x)$ without the use of calculus. 1

End of Question 13

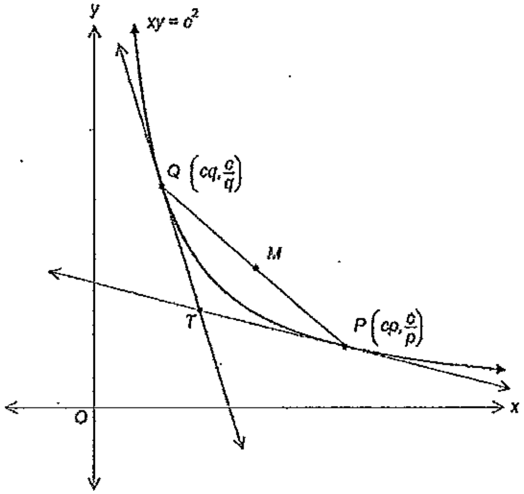
Question 14 (15 marks)

Marks

Start each question on a NEW sheet of paper.

(a) Find the parametric equations which represent the ellipse that passes through $(8,3)$ and has foci at $(-5\sqrt{3}, 0)$ and $(5\sqrt{3}, 0)$. 4

(b) The variable points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the hyperbola $xy = c^2$ as shown in the diagram below.



The tangents to the hyperbola at P and Q intersect at the point T.

M is the midpoint of PQ.

The equation of the tangent at P is $x + p^2 y = 2cp$.

(i) Show that the coordinates of T are $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. 2

(ii) If O is the origin, show that O, T and M are collinear. 2

(iii) Find an expression for q in terms of p if T, M and S are collinear, where S is a focus of the hyperbola. 2

Question 14 continues on the next page.

Question 14 (continued)

Marks

(c) (i) By the use of De Moivre's Theorem, or otherwise, show that 2

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}.$$

(ii) Hence, find expressions for the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$. 3

End of Question 14

Question 15 (15 marks)

Marks

Start each question on a NEW sheet of paper.

(a) Consider $I_n = \frac{1}{n!}\int_0^1 x^n e^{-x}\,dx$, where $n \geq 0$.

(i) Show that $\frac{1}{n!} = e(I_{n-1} - I_n)$. 2

(ii) Find the value of I_4. 2

(b) Evaluate $\int_3^4 \frac{16\,dx}{16-(x-3)^4}$. 4

(c) (i) Show that the equation $x^3 + 5x - 20 = 0$ has exactly one real root, $x = \alpha$, and that $2 < \alpha < 4$. 2

(ii) If $x = \beta$ is one of the other roots of the equation $x^3 + 5x - 20 = 0$, show that 3

$$\sqrt{5} < |\beta| < \sqrt{10}.$$

(d) Solve $\sin 2x + 1 = \sin x + \cos x$ for $0 \leq x \leq 2\pi$. 2

End of Question 15

Question 16 (15 marks)

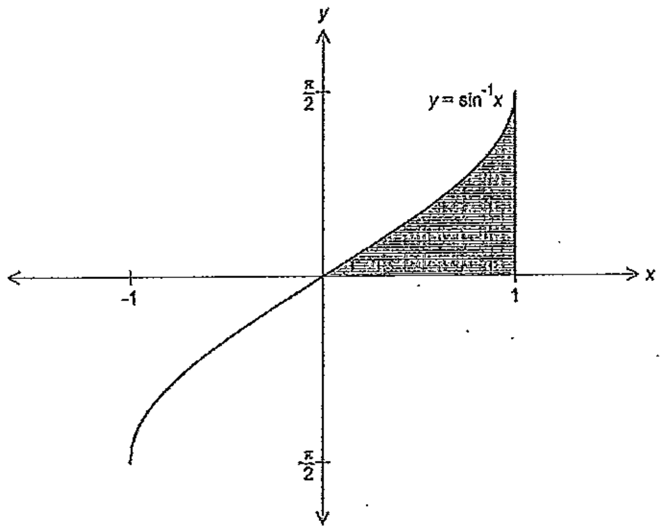
Marks

Start each question on a NEW sheet of paper.

(a)

(i) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx$ using the substitution $x = \sin\theta$. 2

(ii) The area bounded by $y = \sin^{-1} x$, the x −axis and the line $x = 1$ (as shown below) is rotated about the y −axis. Use the method of cylindrical shells to determine the volume of the solid generated. 3



Question 16 continues on the next page.

Question 16 (continued)

Marks

(b)

(i) Show that $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$. 1

(ii) It can also be proven that :

$\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $(x-iy)^3 = x^3 - 3xy^2 + i(y^3 - 3x^2y)$.

(Do NOT prove these results.)

Using the results above, or otherwise, find all real numbers x and y satisfying: 4

$$\left.\begin{aligned} x^3 - 3xy^2 &= 1 \\ y^3 - 3x^2y &= 1 \end{aligned}\right\}$$

Express your answers in surd form.

(c) A cycling track contains a bend that is part of a circle of radius 12 m. At the bend, the track is banked at an angle 30° to the horizontal. A bicycle of mass m kg travels around the bend at constant speed v. Assume that the forces acting on the bicycle are the gravitational force mg, a sideways frictional force F and a normal reaction N to the track.

(i) Resolve the forces acting on the bicycle into their horizontal and vertical components. 2

(ii) The maximum frictional force (up or down the track) is at most $\frac{1}{10}$ of the normal reaction force. 3

Find the range of speeds at which the bicycle can travel safely around the bend. Give your answers correct to 2 decimal places. (Assume that the value of g is 9.8ms^{-2}.)

End of paper

Sydney Girls High School
Mathematics Faculty

Multiple Choice Answer Sheet
2016 Trial HSC Mathematics Extension 2

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ~~●~~ B ~~●~~ C ○ D ○

Student Number: ANSWERS.

Completely fill the response oval representing the most correct answer.

1. A ○ B ● C ○ D ○
2. A ○ B ○ C ○ D ●
3. A ○ B ○ C ○ D ●
4. A ○ B ○ C ● D ○
5. A ○ B ● C ○ D ○
6. A ● B ○ C ○ D ○
7. A ○ B ○ C ○ D ●
8. A ○ B ○ C ○ D ●
9. A ○ B ○ C ○ D ●
10. A ○ B ○ C ● D ○

SGHS TRIAL 2016: SOLUTIONS (Extension 2)

1. $\alpha + \beta + \gamma = \frac{12}{24}$ $\alpha + \alpha = \frac{1}{2}$ $\therefore \alpha = \frac{1}{4}$ (B)

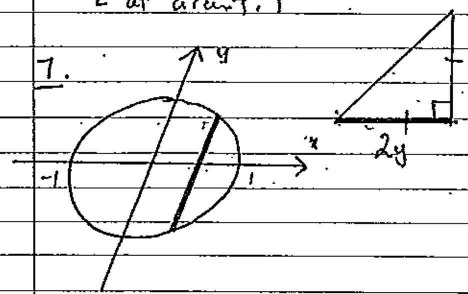
2. $\int \frac{dx}{(x-2)^2 + 9} = \frac{1}{3} \tan^{-1}\left(\frac{x-2}{3}\right) + c$ (D)

3. $(r \operatorname{cis} \theta)^2 = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ $2\theta = \frac{\pi}{4} + 2k\pi$ where k is an integer.
$\theta = \frac{\pi}{8} + k\pi$
$\therefore \theta = \frac{\pi}{8}, \frac{9\pi}{8} \left(= \frac{-7\pi}{8}\right)$ (D)

4. $r = 4$ cm $V = 80$ cm/min
$V = r\omega$ $\therefore \omega = \frac{80}{4} = 20$ rad/min
$= \frac{20}{2\pi}$ revs/min $= \frac{10}{\pi}$ revs/min. (C)

5. $\frac{x^2}{9} - \frac{y^2}{16} = 1$ $ae = 5$ $\frac{a}{e} = \frac{3}{\left(\frac{5}{3}\right)} = \frac{9}{5}$
$e = \frac{5}{3}$
(B)
$\therefore x = \frac{9}{5}$ and $x = -\frac{9}{5}$

6. let $\angle ABC = x$
$\angle ADC = 2x$ $2x + 20 = x + 52$ (considering $\angle$ sum of Δ and vert. opp. $\angle$s) (A)
($\angle$ at centre is twice $\angle$ at circumf.) $x = 32$.

7. 

$A = \frac{1}{2}(2y)^2 = 2y^2$

$V = \int_{-1}^{1} 2y^2 \, dx$ (D)
$= 2\int_{-1}^{1} (1 - x^2) \, dx$

Multiple choice (continued)

8. $\overrightarrow{BA} = i \times \overrightarrow{BC}$

$z - w = i\left(4 + 8i - w\right)$

$a + ib - w = 4i - 8 - iw$

$w\left(1 - i\right) = a + 8 + i\left(b - 4\right)$

$w = \dfrac{a + 8 + i\left(b-4\right)}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{a + 8 - \left(b - 4\right) + i\left(b - 4 + a + 8\right)}{1+1}$

$\therefore\ w = \dfrac{\left(a - b + 12\right) + i\left(a + b + 4\right)}{2}$ (D)

9. # of numbers possible $= \underset{\text{Tom picks}}{20} \times \underset{\text{Joanne picks a different number}}{19}$

Tom can pick any number from 1 to 20 except 10 and 20. Then there will be exactly one different number that Joanne picks to make a total of 20.

$\therefore P(\text{a sum of } 20) = \dfrac{18}{20 \times 19} = \dfrac{9}{190}$ (D)

10. $\displaystyle\lim_{N \to \infty} \sum_{r=1}^{N} \frac{1}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{1\times 2} - \cancel{\dfrac{1}{2\times 3}}\right] + \dfrac{1}{2}\left[\cancel{\dfrac{1}{2\times 3}} - \cancel{\dfrac{1}{3\times 4}}\right] + \dfrac{1}{2}\left[\cancel{\dfrac{1}{3\times 4}} - \dfrac{1}{4\times 5}\right]$

$+ \ldots$

$= \displaystyle\lim_{N \to \infty}\left(\frac{1}{2}\left(\frac{1}{2} - \frac{1}{(N+1)(N+2)}\right)\right)$ (C)

$= \dfrac{1}{2}\left(\dfrac{1}{2} - 0\right) = \dfrac{1}{4}$

SECTION II: Question 11

(a) $z = 2 + 5i$, w [illegible]

(i) $z + w = 2 + 8i$ (1)

(ii) $|zw| = |6i - 15|$

$= \sqrt{6^2 + (-15)^2}$ (1)

A variety of careless answers on this question. Surpisingly, some didn't go further than |6i-15|.

(iii) $\arg\left(w^{71}\right) = 71\arg(w)$

$= 71 \times \dfrac{\pi}{2}$ (1)

$= -\dfrac{\pi}{2}$

A range of confused responses to this question. Some wrote arg (w^71) as arg(71pi/2) which is incorrect. Note the preferred answer was - pi/2.

(b) $(3+i)z^3 + (2i - 19)z^2 + (5 + 2i)z - 3i = 0$

$\alpha + \beta + \gamma = \dfrac{-(2i - 19)}{3+i}$ (1)

$= \dfrac{19 - 2i}{3+i} \times \dfrac{3-i}{3-i}$ (1)

$= \dfrac{57 - 19i - 6i - 2}{9 - i^2}$ (1)

$= \dfrac{11}{2} - \dfrac{5}{2}i$

Calculation errors were common including 2i^2 = 2.

(c) (i) $\displaystyle\int \frac{x}{x^2 + 6x + 25}\,dx = \frac{1}{2}\int \frac{2x+6}{x^2+6x+25}\,dx - 3\int \frac{dx}{(x+3)^2 + 16}$ (1)

$= \dfrac{1}{2}\ln\left(x^2 + 6x + 25\right) - \dfrac{3}{4}\tan^{-1}\left(\dfrac{x+3}{4}\right) + C$ (1)

The key issues with this question involved errors with the 1/2 and mishandling of the minus signs.

(ii) let $u = e^{-x} + 5$ $\quad du = -e^{-x}\,dx$ (1)

$e^{-x} = u - 5$

$I = \displaystyle\int \frac{-(u-5)}{u}\,du = \int\left(-1 + \frac{5}{u}\right)du$ (1)

$= -u + 5\ln u + C$

$\therefore I = -e^{-x} + 5\ln\left(e^{-x} + 5\right) + C$ (1)

Most students selected an appropriate substitution but had varying degrees of success from that point onwards. Common mistakes involved the signs and the splitting of the fraction that needed to be integrated.

Question 11 (continued)

(d) (i) $\sin(A+B) + \sin(A-B)$

$= \sin A\cos B + \sin B\cos A + \sin A\cos B - \sin B\cos A$

$= 2\sin A\cos B$

(ii) $\sin 10x\cos 7x = \dfrac{\sin(10x+7x) + \sin(10x-7x)}{2}$

$= \dfrac{\sin 17x + \sin 3x}{2}$

$\int \sin 10x\cos 7x\,dx = \frac{1}{2}\int(\sin 17x + \sin 3x)\,dx$

$= \frac{1}{2}\left(-\frac{\cos 17x}{17} - \frac{\cos 3x}{3}\right) + c$

$= -\frac{\cos 17x}{34} - \frac{\cos 3x}{6} + c$

Most students did well on this question. However, the occasional student did not make the connection between parts (i) and (ii). Also, there were a variety of responses which involved incorrectly differentiating sin(17x) and sin(3x) instead of integrating.

12 (a) $|z| < 5$

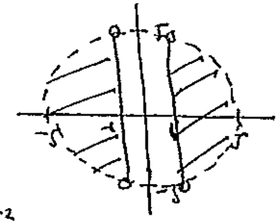
$x^2+y^2 < 25$

$|z+\bar{z}| \geq 2$

$|2x| \geq 2$

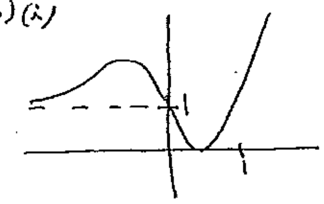
$2x \geq 2$ or $2x \leq -2$

$x \geq 1$ $\quad x \leq -1$

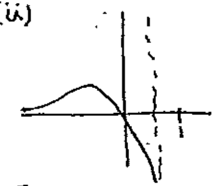


[Many students missed $x \leq -1$]

(b) (i)

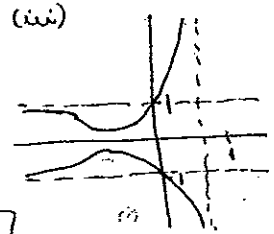


(ii)



[Many students forgot the vertical asymptote]

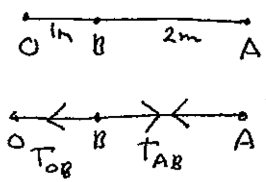
(iii)



$y = \frac{1}{f(x)}$ if $y \geq 0$

$= -\frac{1}{f(x)}$ if $y < 0$

(c)



$T_{AB} = 3 \times 3 \times (2\pi)^2$

$= 36\pi^2$

$T_{OB} - T_{AB} = 5 \times 1 \times (2\pi)^2$

$= 20\pi^2$

$\therefore T_{OB} = 36\pi^2 + 20\pi^2$

$= 56\pi^2$

$T_{OB} : T_{AB} = 56 : 36$

$= 14 : 9$

(d) $\dfrac{BN}{\sin B\hat{A}N} = \dfrac{AB}{\sin B\hat{N}A}$ $\quad \dfrac{CN}{\sin N\hat{A}C} = \dfrac{AC}{\sin A\hat{N}C}$

$\dfrac{\sin B\hat{N}A}{\sin B\hat{A}N} = \dfrac{AB}{BN}$ $\quad \dfrac{\sin A\hat{N}C}{\sin N\hat{A}C} = \dfrac{AC}{CN}$

$B\hat{A}N = N\hat{A}C$ (given)

$B\hat{N}A = 180° - A\hat{N}C$ (straight $\angle$)

$\therefore \sin B\hat{A}N = \sin N\hat{A}C$

$\sin B\hat{N}A = \sin A\hat{N}C$

$\therefore \dfrac{AB}{BN} = \dfrac{AC}{CN}$

[Δs ABN and ANC are not similar]

2016 THSC Ext 2

Q13

(a) $x^2+xy+2y^2=28$

$2x+y+xy'+4yy'=0.$

$y'(x+4y)=-2x-y$

$y'=\frac{-2x-y}{x+4y}.$

At $(-2,-3)$ $m_T=\frac{4+3}{-2-12}$

$m_T=-\frac{1}{2}.$

$m_2=1$

$\tan\theta=\left|\frac{m_T-m_2}{1+m_Tm_2}\right|$

$\tan\theta=\left|\frac{-\frac{1}{2}-1}{1+(-\frac{1}{2})(1)}\right|$

$=\left|\frac{-\frac{3}{2}}{\frac{1}{2}}\right|$

$=3$

$\theta=\tan^{-1}(3)\approx 71^\circ 34'=72^\circ$

Many students lost a mark for not using the formula for angle between lines correctly.

(b)

(i) $x=\alpha,\beta,\gamma$

Let $u=x^2+4.$

So $x=\sqrt{u-4}$

Sub into $x^3-5x^2+7x-18=0$

$(u-4)\sqrt{u-4}-5(u-4)+7\sqrt{u-4}-18=0.$

$\sqrt{u-4}(u+3)=18-5u-20.$

$(u-4)(u+3)^2=(5u-2)^2$

$(u-4)(u^2+6u+9)=25u^2-20u+4$

$u^3+6u^2+9u-4u^2-24u-36=25u^2-20u+4.$

$u^3-23u^2+5u-40=0$

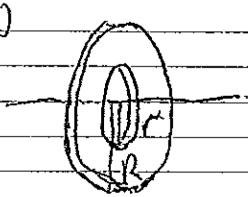
(ii) Sum of roots

$\alpha^2+\beta^2+\gamma^2+12=23$

$\alpha^2+\beta^2+\gamma^2=11.$

Many students lost marks due to poor algebra skills.

(c)



$$\delta V = (\pi R^2 - \pi r^2)\delta x$$
$$= \pi\left((20-y)^2 - 15^2\right)\delta x$$
$$= \pi(175 - 40y + y^2)\delta x$$

Intersect $x^2 = 20y$ and $y = 5$.

$$= \pi\left(175 - 2x^2 + \frac{x^4}{400}\right)\delta x$$

$$x^2 = 100$$
$$x = \pm 10.$$

$$V = \lim_{\delta x \to 0} \sum_{x=-10}^{10} \pi\left(175 - 2x^2 + \frac{x^4}{400}\right)\delta x$$
$$= 2\pi \int_0^{10} \left(175 - 2x^2 + \frac{x^4}{400}\right) dx$$
$$= 2\pi \left[175x - \frac{2x^3}{3} + \frac{x^5}{2000}\right]_0^{10}$$
$$= 2\pi\left(1750 - \frac{2000}{3} + 50\right)$$
$$= \frac{6800\pi}{3} \text{ units}^3$$

> Most student attempted to use the shells method for this question. Only a few were able to do that correctly.

(d) $P(x) = 8x^4 + 28x^3 + 18x^2 - 27x - 27$

(i) $P'(x) = 32x^3 + 84x^2 + 36x - 27$

$P''(x) = 96x^2 + 168x + 36$

Possible roots where $P''(x) = 0$.

$$8x^2 + 14x + 3 = 0$$
$$\frac{(8x+12)(8x+2)}{8} = 0$$
$$(2x+3)(4x+1) = 0.$$
$$x = -\frac{3}{2}, -\frac{1}{4}$$

$P(-\frac{1}{4}) \neq 0$ $\quad P(-\frac{3}{2}) = 0.$

So triple root when $x = -\frac{3}{2}$.

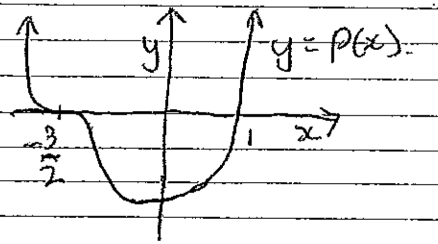
That is $(2x+3)^3(x-b) = 0.$

Constant term

$$-27b = -27$$
$$b = 1$$

Zeros are $-\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}$ and 1

(ii)



Q14 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through $(8,3)$

a)

$$\frac{64}{a^2}+\frac{9}{b^2}=1 \quad (1)$$

Also $b^2=a^2(1-e^2)$

$b^2=a^2-a^2e^2$ $\quad (ae=5\sqrt{3})$

$a^2=b^2+75 \quad (2)$

Subs (2) into (1): $\frac{64}{b^2+75}+\frac{9}{b^2}=1$

$64b^2+9b^2+675=b^4+75b^2$

$b^4+2b^2-675=0$

Let $u=b^2$

$u^2+2u-675=0$

$u=-27$ (Reject)

$u=25 \longrightarrow b=5 \quad (b>0)$

$a^2=75+b^2$

$a^2=100$

$a=10$

Thus the Equation of the Ellipse in the parametric form:

$x=10\cos\theta$

$y=5\sin\theta$

Alternatively: $PS+PS'=2a$ can also be used to find a.

Some students wrote $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as the Eq of the Ellipse which gave the wrong values for a and b.

Q14

b) Eq of the tangent at P : $x^2 + p^2 y = 2cp$ ①

i) Eq ------ at Q : $x^2 + q^2 y = 2cq$ ②

① − ② : $(p+q)(p-q)y = 2c(p-q)$

$$y = \frac{2c}{p+q} \text{ subs into ①}$$ ✓

$$x = 2cp - p^2\left(\frac{2c}{p+q}\right)$$

$$x = \frac{2cpq}{p+q}$$ ✓

$$\therefore T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

ii) M, midpoint of PQ

$$M\left(\frac{cp+cq}{2}, \frac{c/p + c/q}{2}\right)$$

$$M\left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right)$$

If O, T, M are collinear if

$$mOT = mTM$$

$$mOT = \frac{2c/p+q}{\frac{2cpq}{p+q}} = \frac{1}{pq}$$ ✓

$$mTM = \frac{\frac{c(p+q)}{2pq} - \frac{2c}{p+q}}{\frac{c(p+q)}{2} - \frac{2cpq}{p+q}} = \frac{2(p+q)}{2pq(p+q)} = \frac{1}{pq}$$ ✓

$\therefore$ O, T, M are collinear.

Most students did well in these two parts.

Q14

b/iii) Focus point $S(\sqrt{2}c, \sqrt{2}c)$ lies on TM

Eq of TM : $y - \frac{2c}{p+q} = \frac{1}{pq}\left(x - \frac{2cpq}{p+q}\right)$

$$\sqrt{2}c - \frac{2c}{p+q} = \frac{1}{pq}\left(\sqrt{2}c - \frac{2cpq}{p+q}\right)$$ ✓

$$2\sqrt{c} - \frac{2c}{p+q} = \frac{\sqrt{2}c}{pq} - \frac{2c}{p+q}$$

$$2\sqrt{c}\, pq = \sqrt{2}c$$

$$pq = 1 \quad \text{OR} \quad q = \frac{1}{p}$$ ✓

Some students could not regconise the focus point is $S(\sqrt{2}c, \sqrt{2}c)$ and it lies on TM.

Q14

c/i)

$$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3 i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$$

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4\cos^3\theta\, i\sin\theta - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta\, i + \sin^4\theta$$

$$\cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$$ ✓

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$ ✓

Divide by $\cos^4\theta$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan\theta + 1}$$

Q14

c(ii)

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}$$

Let $x = \tan\theta$

$$\frac{4x - 4x^3}{x^4 - 6x^2 + 1} = 1$$

OR $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

OR $\tan 4\theta = 1$

$$4\theta = \frac{\pi}{4} + k\pi$$

$$\theta = \frac{\pi(1 + 4k)}{16}$$

$$x = \tan\left[\frac{(4k+1)\pi}{16}\right]$$

where $k = 0, 1, 2, 3$

OR $x = \tan\frac{\pi}{16},\ \tan\frac{5\pi}{16},\ \tan\frac{9\pi}{16},\ \tan\frac{13\pi}{16}$

A number of students could not deduce $\tan 4\theta = 1$ or could not provide the final solutions for x.

15 (a) (i) Let $u = x^n$ $\quad v' = e^{-x}$

$u' = nx^{n-1}$ $\quad v = -e^{-x}$

$$I_n = \frac{1}{n!}\left\{\left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx\right\}$$

$$= \frac{1}{n!}\left\{-e^{-1} + n\int_0^1 x^{n-1}e^{-x}\,dx\right\}$$

$$= -\frac{1}{en!} + \frac{1}{(n-1)!}\int_0^1 x^{n-1}e^{-x}\,dx$$ [Many students made errors at this point]

$$= -\frac{1}{en!} + I_{n-1}$$

$$\frac{1}{en!} = I_{n-1} - I_n$$

$$\frac{1}{n!} = e\left(I_{n-1} - I_n\right)$$

(ii)

$$I_4 = I_3 - \frac{1}{e.4!} = I_3 - \frac{1}{24e} = 1 - \frac{8}{3e} - \frac{1}{24e} = 1 - \frac{65}{24e}$$

$$I_3 = I_2 - \frac{1}{e.3!} = I_2 - \frac{1}{6e} = 1 - \frac{5}{2e} - \frac{1}{6e} = 1 - \frac{16}{6e} = 1 - \frac{8}{3e}$$

$$I_2 = I_1 - \frac{1}{e.2!} = I_1 - \frac{1}{2e} = 1 - \frac{2}{e} - \frac{1}{2e} = 1 - \frac{5}{2e}$$

$$I_1 = I_0 - \frac{1}{e} = 1 - \frac{1}{e} - \frac{1}{e} = 1 - \frac{2}{e}$$

$$I_0 = \int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}$$

(b) Let $u = x - 3$

$$\frac{du}{dx} = 1$$

$$du = dx$$

when $x = 4$, $u = 1$

" $x = 3$, $u = 0$

$$\int_0^1 \frac{16}{16 - u^4}\,du$$ [The primitive of this is not a log function]

$$= \int_0^1 \left(\frac{2}{4 - u^2} + \frac{2}{4 + u^2}\right) du$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{2-u} + \frac{1}{2+u}\right) du + \int_0^1 \frac{2}{4+u^2}\,du$$

$$= \frac{1}{2}\left[-\log(2-u) + \log(2+u)\right]_0^1 + \left[\tan^{-1}\frac{u}{2}\right]_0^1$$

$$= \frac{1}{2}\left(-\log 1 + \log 3 + \log 2 - \log 2\right) + \tan^{-1}\frac{1}{2}$$

$$= \frac{1}{2}\log 3 + \tan^{-1}\frac{1}{2}$$

(c) (i) $y' = 3x^2 + 5$

$3x^2+5=0$ has no real solutions

$\therefore$ no turning points

$\therefore$ 1 real root

$2^3 + 5\times 2 - 20 = -2$

$4^3 + 5\times 4 - 20 = 64$

$\therefore\ 2 < \alpha < 4$

(ii) β is a complex root

$\therefore\ \bar{\beta}$ is also a " "

$\alpha\beta\bar{\beta} = \alpha|\beta|^2 = -\frac{-20}{1}$
$= 20$

$\alpha = \frac{20}{|\beta|^2}$

$2 < \frac{20}{|\beta|^2} < 4$

$\frac{1}{4} < \frac{|\beta|^2}{20} < \frac{1}{2}$

$5 < |\beta|^2 < 10$

$\sqrt{5} < |\beta| < \sqrt{10}$

(d) $2\sin x\cos x + \sin^2 x + \cos^2 x = \sin x + \cos x$

$(\sin x + \cos x)^2 = \sin x + \cos x$

$(\sin x + \cos x)^2 - (\sin x + \cos x) = 0$

$(\sin x + \cos x)(\sin x + \cos x - 1) = 0$

$\sin x + \cos x = 0$

$\sin x = -\cos x$

$\tan x = -1$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$\sin x + \cos x = 1$

$A = \sqrt{1+1}$ $\quad \tan\alpha = \frac{1}{1}$

$= \sqrt{2}$ $\quad \alpha = \frac{\pi}{4}$

$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1$

$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$

$x = 0, \frac{\pi}{2}, 2\pi$

[Students needed to have at least one correct solution to get a mark.]

$\therefore\ x = 0, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{7\pi}{4}, 2\pi$

Question 16

(a) (i) let $x = \sin\theta$ $\quad dx = \cos\theta\, d\theta$

$I = \int_{\sin^{-1}0}^{\sin^{-1}1} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$ 1 since $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta}$
$= \cos\theta$

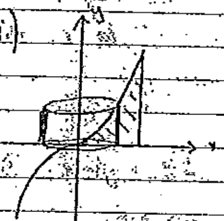
$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2\theta)\, d\theta$ $\quad \cos 2\theta = 1 - 2\sin^2\theta$
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sin\pi}{2} - (0)\right]$ 1 $\therefore\ I = \frac{\pi}{4}$

The better solutions converted the limits to reflect the θ values. ~~It was important to use~~ the correct relationship between sin (2θ) and cos(θ).

(ii)



$V_{shell} = \pi R^2 h - \pi r^2 h$ $\quad h = y$, $R = x + \delta x$, $r = x$

$= \pi y(x + \delta x + x)(x + \delta x - x)$

$= \pi y(2x + \delta x)\delta x$

$= 2\pi x y\,\delta x$ as δx^2 is negligible ($\delta x \to 0$)

$V_{solid} = 2\pi\int_0^1 xy\, dx$ 1

$= 2\pi\int_0^1 x\sin^{-1}x\, dx$

BY PARTS $\quad u = \sin^{-1}x \quad v' = x$
$u' = \frac{1}{\sqrt{1-x^2}} \quad v = \frac{x^2}{2}$

$\therefore\ V_{solid} = 2\pi\left(\left[\frac{x^2}{2}\sin^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx\right)$ 1

$= 2\pi\left(\frac{1}{2}\sin^{-1}1 - 0 - \frac{1}{2}\times\frac{\pi}{4}\right)$

$= 2\pi\left(\frac{\pi}{4} - \frac{\pi}{8}\right)$ 1

$= 2\pi \times \frac{\pi}{8}$

$\therefore\ V = \frac{\pi^2}{4}$ units^3

The 2π at the front vanished in some solutions. ~~Critically, some students did not make~~ the connection to their answer from part (i). This meant that time was wasted on evaluating the integral that ~~they had already calculated.~~

16 (continued)

Part (i) should have been manageable. A number of students attempted to "show" by writing down the decimal approximations to the left and right side from their calculator. This was not accepted for (hopefully) obvious reasons.

(b) (i) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ (1)

The vast majority of students found part (ii) difficult and failed to make the connection between the simultaneous equations and the (x-iy)^3 relationship. This was a critical first step.

(ii) $\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$ $\quad (x-iy)^3 = x^3 - 3xy^2 + i(y^3 - 3x^2y)$

If $x^3 - 3xy^2 = 1$ and $y^3 - 3x^2y = 1$ then $(x-iy)^3 = 1+i$ (1)

Let $x - iy = r\,\text{cis}\,\theta$ $\therefore (r\,\text{cis}\,\theta)^3 = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

By De Moivre's Theorem $r^3 = \sqrt{2}$ $\quad 3\theta = \frac{\pi}{4} + 2k\pi$ where k is an integer

$\therefore r = \sqrt[6]{2}$ $\quad \theta = \frac{\pi + 8k\pi}{12} = \frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}$ (1)

Case ① $x - iy = \sqrt[6]{2}\,\text{cis}\,\frac{\pi}{12} = \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$\therefore x = \sqrt[6]{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$ $\quad y = -\sqrt[6]{2}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$ (1)

Case ② $x - iy = \sqrt[6]{2}\,\text{cis}\,\frac{3\pi}{4}$

$\therefore x = \sqrt[6]{2} \times -\frac{1}{\sqrt{2}} = -\frac{\sqrt[6]{2}}{\sqrt{2}}$ $\quad y = -\sqrt[6]{2} \times \frac{1}{\sqrt{2}} = -\frac{\sqrt[6]{2}}{\sqrt{2}}$

Case ③ $x - iy = \sqrt[6]{2}\,\text{cis}\,\frac{17\pi}{12} = \sqrt[6]{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

$x = \sqrt[6]{2}\cos\left(\frac{3\pi}{2} - \frac{\pi}{12}\right) = \sqrt[6]{2}\left(\cos\frac{3\pi}{2}\cos\frac{\pi}{12} + \sin\frac{3\pi}{2}\sin\frac{\pi}{12}\right)$

$= -\sqrt[6]{2}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$ (1)

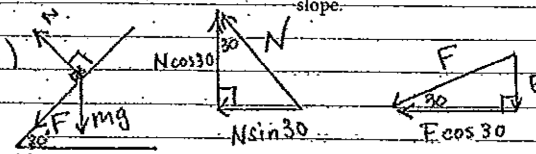
$y = -\sqrt[6]{2}\sin\left(\frac{3\pi}{2} - \frac{\pi}{12}\right) = -\sqrt[6]{2}\left(\sin\frac{3\pi}{2}\cos\frac{\pi}{12} - \cos\frac{3\pi}{2}\sin\frac{\pi}{12}\right)$

$= +\sqrt[6]{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$

16 (continued)

Students are recommended to resolve the forces with the assistance of clear diagrams. Note that in this part, the force F could be drawn either up or down the slope.

(c) (i)



(1) for diagrams.

Note: If F is directed up the slope, the diagram for F and the results below will be slightly different.

Horiz. $N\sin 30 + F\cos 30 = \frac{mv^2}{r}$ ①

Vert. $N\cos 30 - F\sin 30 = mg$ ②

(1) for both correct.

(ii) ① × sin 30 + ② × cos 30 $\quad N = \frac{mv^2}{r}\sin 30 + mg\cos 30$

(1) for either

① × cos 30 − ② × sin 30 $\quad F = \frac{mv^2}{r}\cos 30 - mg\sin 30$ ③

Since $F \le \frac{N}{10}$ then $10F \le N$

i.e. $10\left(\frac{mv^2}{12} \times \frac{\sqrt{3}}{2} - mg \times \frac{1}{2}\right) \le \frac{mv^2}{12} \times \frac{1}{2} + mg \times \frac{\sqrt{3}}{2}$

$10\sqrt{3}\,mv^2 - 120mg \le mv^2 + 12\sqrt{3}\,mg$

$\cancel{m}v^2(10\sqrt{3} - 1) \le \cancel{m}g(120 + 12\sqrt{3})$ (1) or

$v^2 \le \frac{g(120 + 12\sqrt{3})}{10\sqrt{3} - 1}$

$\therefore v \le 9.19$

Many students did not get the question out completely. Most solutions failed to consider that the frictional force could be directed either up or down the slope. The fastest speed was determined using the downwards version and the upwards version was needed for the slowest possible speed.

For minimum speed, consider when F is acting up the slope.

In this case (using ③) $F = -\left(\frac{mv^2}{r}\cos 30 - mg\sin 30\right)$

Using similar algebra to the above (where $10F \le N$).

$-10\sqrt{3}\,mv^2 + 120mg \le mv^2 + 12\sqrt{3}\,mg$

$\cancel{m}g(120 - 12\sqrt{3}) \le \cancel{m}v^2(1 + 10\sqrt{3})$

$\therefore v^2 \ge \frac{g(120 - 12\sqrt{3})}{1 + 10\sqrt{3}}$ $\quad \therefore v \ge 7.29$

$\therefore$ The bike can travel speeds from 7.29 m/s to 9.19 m/s. (1)