CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Half-Yearly Examination

# FORM VI

# MATHEMATICS 2 UNIT

Thursday 25th February 2016

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.

Total — 60 Marks

- All questions may be attempted.

Section I – 8 Marks

- Questions 1–8 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 52 Marks

- Questions 9–12 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Nine.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 87 boys

Examiner
LRP



SECTION I - Multiple Choice

**Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.**

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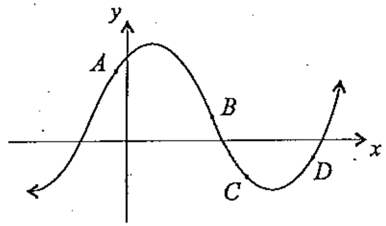
QUESTION ONE

1




The graph of $y = f(x)$ is shown above. At which point is $f'(x) > 0$ and $f''(x) > 0$?

(A) $A$
(B) $B$
(C) $C$
(D) $D$

QUESTION TWO

What is the focal length of the parabola $x^2 = 4y$? 1

(A) $\frac{1}{4}$
(B) 1
(C) 4
(D) 8

QUESTION THREE

What is the domain of the curve $y = \log_e(x-1)$? 1

(A) $x > -1$
(B) $x > 0$
(C) $x > 1$
(D) all real $x$

## QUESTION FOUR

Simpson's Rule is used with three function values to approximate $\int_1^2 xe^x\,dx$. Which expression is obtained? 1

(A) $\frac{1}{2}\left(e+3e^{1\cdot5}+2e^2\right)$

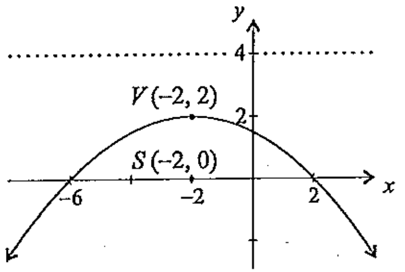
(B) $\frac{1}{4}\left(e+3e^{1\cdot5}+2e^2\right)$

(C) $\frac{1}{6}\left(e+6e^{1\cdot5}+2e^2\right)$

(D) $\frac{1}{12}\left(e+6e^{1\cdot5}+2e^2\right)$

## QUESTION FIVE

1




A parabola with focus $S(-2,0)$ and directrix $y=4$ is shown above. Which of the following is the equation of the parabola?

(A) $(x+2)^2=-8(y-2)$

(B) $(x+2)^2=-8y$

(C) $(x-2)^2=-8(y+2)$

(D) $(x-2)^2=-8y$





## QUESTION SIX

What is the value of $k$ if $\int_1^k \frac{2}{x}\,dx=4$? 1

(A) $e^2$

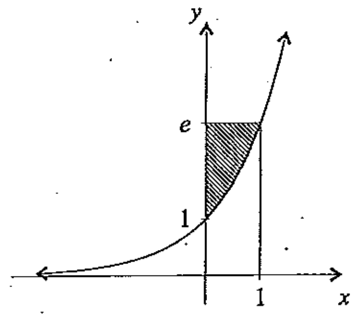
(B) $e^3$

(C) $e^4$

(D) $e^5$

## QUESTION SEVEN

1




The graph of $y=e^x$ is shown above. Which of the following expressions does NOT represent the area of the shaded region?

(A) $e-\int_0^1 e^x\,dx$

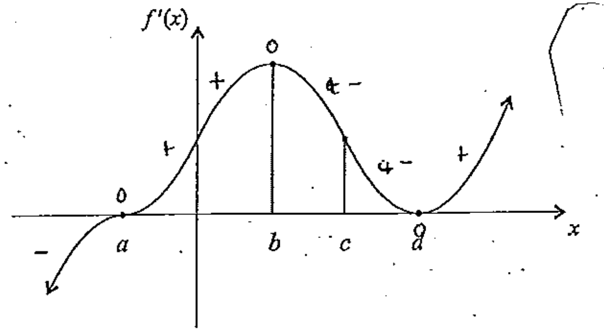
(B) $\int_1^e e^x\,dx$

(C) $\int_1^e \log_e y\,dy$

(D) $\int_0^1 (e-e^x)\,dx$

## QUESTION EIGHT

1




The graph of the gradient function $y = f'(x)$ is shown above. Which of the labelled $x$-values corresponds to a stationary point of inflexion on the original curve $y = f(x)$?

(A) $a$

(B) $b$

(C) $c$

(D) $d$

——— End of Section I ———





## SECTION II - Written Response

**Answers for this section should be recorded in the booklets provided.**

**Show all necessary working.**

**Start a new booklet for each question.**

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## QUESTION NINE (13 marks) Use a separate writing booklet.

Marks

(a) Calculate $\dfrac{3}{e^2}$ correct to 3 significant figures. 1

(b) Expand and simplify $e^x\left(e^x + e^{-x}\right)$. 1

(c) Differentiate:

(i) $\sqrt{x}$ 1

(ii) $e^{2x}$ 1

(iii) $\log_e 3x$ 1

(d) Find a primitive of:

(i) $6x^2$ 1

(ii) $e^{4x}$ 1

(iii) $\dfrac{3}{x}$ 1

(e) Evaluate $\displaystyle\int_2^3 (2x-1)\,dx$. 2

(f) Find the equation of the parabola with vertex $(0,0)$ and focus $(4,0)$. 1

(g) Consider the curve whose first derivative is given by $y' = x^2 + x - 6$. For what values of $x$ is the curve increasing? 2

QUESTION TEN (13 marks) Use a separate writing booklet. Marks

(a) Differentiate the following with respect to $x$:

(i) $y = \log_e(x^2 + 3)$ [1]

(ii) $y = \dfrac{x+1}{x}$ [2]

(b) Find the equation of the tangent to the curve $y = e^{3x} + 1$ at the point $(0, 2)$. [2]

(c) Consider the parabola with equation $y^2 = -8(x + 2)$.

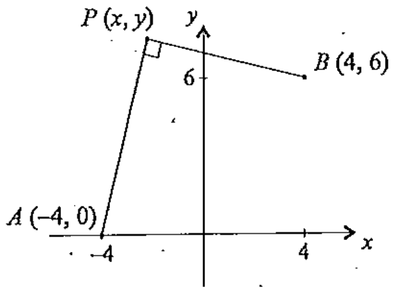
(i) Write down the coordinates of the vertex. [1]

(ii) Find the coordinates of the focus. [1]

(iii) Find the equation of the directrix. [1]

(iv) Sketch the parabola clearly showing the vertex, focus and directrix. [1]

(d)




Consider the points $A(-4, 0)$ and $B(4, 6)$. The point $P(x, y)$ moves so that $PA$ is always perpendicular to $PB$.

(i) Find expressions for the gradients of $PA$ and $PB$. [1]

(ii) Hence show that the equation of the locus of $P$ is $x^2 + y^2 - 6y - 16 = 0$. [1]

(iii) Express the above equation in the form $(x - h)^2 + (y - k)^2 = r^2$. [1]

(iv) Hence, or otherwise, describe the locus of $P$ geometrically. [1]

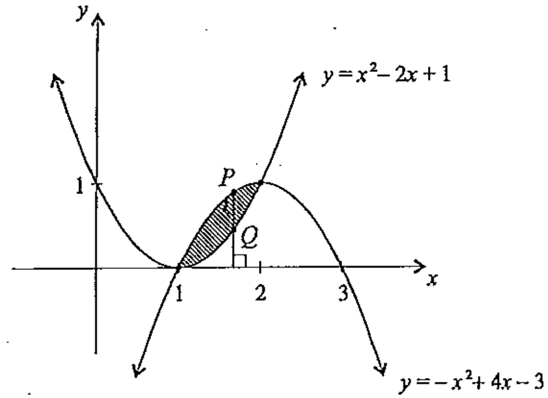
Exam continues overleaf ...



QUESTION ELEVEN (13 marks) Use a separate writing booklet. Marks

(a) If $f'(x) = \dfrac{2x - 3}{x^2 - 3x + 1}$ and $f(3) = 5$, find $f(x)$. [2]

(b)




The diagram above shows the curves $y = -x^2 + 4x - 3$ and $y = x^2 - 2x + 1$.

(i) Show that the curves intersect at the points where $x = 1$ and $x = 2$. [1]

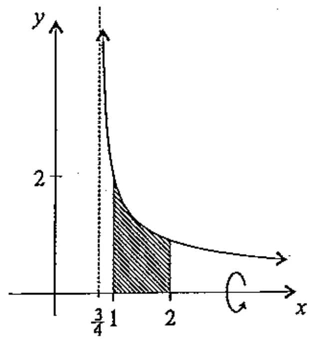
(ii) Find the area of the shaded region enclosed between the two curves. [3]

The points $P$ and $Q$ are located on the curves $y = -x^2 + 4x - 3$ and $y = x^2 - 2x + 1$ respectively, and share the same $x$-coordinate as shown.

(iii) If the length of $PQ$ is $\ell$ units, show that $\ell = -2x^2 + 6x - 4$. [1]

(iv) Hence find the maximum length of $PQ$ for $1 \le x \le 2$. [3]

(c) [3]




The region bounded by the curve $y = \dfrac{3}{\sqrt{4x - 3}}$, the $x$-axis and the lines $x = 1$ and $x = 2$ is shown above. Find the volume of the solid generated when this region is rotated about the $x$-axis.

Exam continues next page ...

QUESTION TWELVE (13 marks) Use a separate writing booklet. **Marks**

(a) Solve for $x$: **2**

$$\log_2\left(x^2 - 2x\right) = 3.$$

(b) (i) Show that the function $y = x\log_e x$ has derivative $\dfrac{dy}{dx} = \log_e x + 1$. **1**

(ii) Hence find $\displaystyle\int \log_e x\,dx$. **2**

(c) Consider the curve with equation $y = \dfrac{x}{e^x}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{1-x}{e^x}$ and $\dfrac{d^2y}{dx^2} = \dfrac{x-2}{e^x}$. **2**

(ii) Find the coordinates of the stationary point and determine its nature. **2**

(iii) Find the coordinates of the point of inflexion. You must show that it is a point of inflexion. **2**

(iv) Sketch the curve, clearly showing the stationary point, the point of inflexion and any intercepts with the coordinate axes. **2**

(You may use the fact that as $x \to \infty$, $\dfrac{x}{e^x} \to 0^+$.)

——— End of Section II ———

END OF EXAMINATION

# MATHEMATICS 2U

## Half-Yearly Examination - SOLUTIONS

### * Multiple Choice

Q1. D   positive gradient
* concave up

Q2. B   $a = 1$

Q3. C   $x - 1 > 0$
$\therefore x > 1$

Q4. C

| $x$ | 1 | 1.5 | 2 |
|---|---|---|---|
| $y$ | $e$ | $\frac{3}{2}e^{1.5}$ | $2e^2$ |

$\frac{1}{6}\left(e + 4 \times \frac{3}{2}e^{1.5} + 2e^2\right)$

Q5. A   $(x-h)^2 = -4a(y-k)$
$(x-(-2))^2 = -4 \times 2(y-2)$

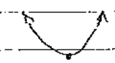
Q6. A   $\int_1^k \frac{2}{x}\,dx = 4 \rightarrow \left[2\ln x\right]_1^k = 4$

$2\ln k - 2\cancel{\ln 1}^{\,0} = 4$
$\ln k = 2$
$\therefore e^2 = k$

Q7. B

Q8. D   for stationary point of inflexion either

+/ ‾‾ 0 /+   or   \‾ 0 ‾‾ \-   on original curve

→ at d :  $f'(x)$ has a +ve value either side of
← 0 at point d.

* QUESTION 9:

a) $\frac{3}{e^2} = 0.4060058\ldots$

$= 0.406$ (to 3 sig. fig.) ✓

b) $e^x(e^x + e^{-x}) = e^{2x} + e^0$

$= e^{2x} + 1$ ✓

c) i) $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$

✓ either.

$= \frac{1}{2\sqrt{x}}$

ii) $\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$ ✓

iii) $\frac{d}{dx}\left(\log_e 3x\right) = \frac{1}{3x} \times 3$

$= \frac{1}{x}$ ✓

d) i) $y' = 6x^2$

$y = \frac{6x^3}{3} (+c)$

$= 2x^3 (+c)$ ✓

ii) $y' = e^{4x}$

$y = \frac{e^{4x}}{4} (+c)$ ✓

iii) $y' = \frac{3}{x}$

$y = 3\ln x (+c)$ ✓

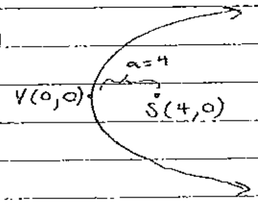
don't penalise missing +c.

e) $\int_2^3 (2x-1)\,dx = \left[x^2 - x\right]_2^3$ ✓

$= 3^2 - 3 - (2^2 - 2)$ ✓

$= 4$

f)



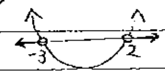

$y^2 = 4ax$

$y^2 = 16x$ ✓

g) increasing $\rightarrow y' > 0$

$x^2 + x - 6 > 0$ ✓

$(x+3)(x-2) > 0$




$x < -3$ or $x > 2$ ✓

/13

* QUESTION 10:

a) i) $\frac{dy}{dx} = \frac{1}{x^2+3} \times 2x$ ✓

$= \frac{2x}{x^2+3}$

ii) $y = \frac{x}{x} + \frac{1}{x}$

$= 1 + x^{-1}$ ✓

$\frac{dy}{dx} = -x^{-2}$ ✓

$= -\frac{1}{x^2}$

OR// Let $u = x+1$ $\quad v = x$

$u' = 1$ $\quad v' = 1$ ✓

$\frac{dy}{dx} = \frac{x(1) - (x+1)(1)}{x^2}$

$= -\frac{1}{x^2}$ ✓

b) $\frac{dy}{dx} = 3e^{3x}$

when $x = 0$, $\frac{dy}{dx} = 3$ ✓

$y - 2 = 3(x - 0)$

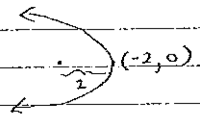
$\therefore y = 3x + 2$ ✓

c) $y^2 = -8(x+2)$

i) Vertex $(-2, 0)$ ✓

ii) $4a = 8$

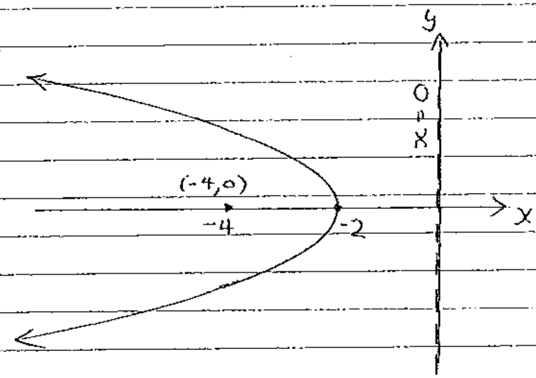
$a = 2$




Focus $(-4, 0)$ ✓

iii) Directrix $x = 0$ ✓

iv)




✓

d) i) $m_{PA} = \frac{y}{x+4}$

$m_{PB} = \frac{y-6}{x-4}$ ✓

ii) $m_{PA} \times m_{PB} = -1$

$\frac{y}{x+4} \times \frac{y-6}{x-4} = -1$ ✓ show that...

$y^2 - 6y = -x^2 + 16$

$\therefore x^2 + y^2 - 6y - 16 = 0$

iii) $x^2 + y^2 - 6y + 9 = 16 + 9$

$x^2 + (y-3)^2 = 25$

$(x-0)^2 + (y-3)^2 = 5^2$

iv) The locus of P is a circle with centre (0,3) & radius 5 units

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* QUESTION 11:

a) $f(x) = \log_e(x^2 - 3x + 1) + C$

$5 = \log_e(9 - 9 + 1) + C$

$5 = \cancel{\log_e 1}^{0} + C$

$\therefore f(x) = \log_e(x^2 - 3x + 1) + 5$

b) i) $x^2 - 2x + 1 = -x^2 + 4x - 3$

$2x^2 - 6x + 4 = 0$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$\therefore x = 1$ or $x = 2$.

Alternatively show the substitutions of both $x$ values into both equations → four substitutions

ii) $A = \int_1^2 \left(-x^2 + 4x - 3 - (x^2 - 2x + 1)\right) dx$

$= \int_1^2 (-2x^2 + 6x - 4)\, dx$

$= -2\int_1^2 (x^2 - 3x + 2)\, dx$

$= -2\left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2$

$= -2\left(\frac{8}{3} - 6 + 4 - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right)$

$= \frac{1}{3}\ u^2$

iii) $l = y_P - y_Q$

$= -x^2 + 4x - 3 - (x^2 - 2x + 1)$ ✓ show that...

$= -2x^2 + 6x - 4$

iv) $\frac{dl}{dx} = -4x + 6$

$= 0$ when $x = \frac{3}{2}$ ✓

$\frac{d^2l}{dx^2} = -4$

$< 0$ $\therefore$ ∩ ✓ $\therefore$ max occurs when $x = \frac{3}{2}$

$\therefore l_{max} = -2\left(\frac{3}{2}\right)^2 + 6\left(\frac{3}{2}\right) - 4$

$= \frac{1}{2}$ units ✓

Alternatively, for middle mark:

| $x$ | $1$ | $\frac{3}{2}$ | $2$ |
|---|---|---|---|
| $\frac{dl}{dx}$ | $2$ | $0$ | $-2$ |
| | / | — | \ |

✓

$\therefore$ max occurs when $x = \frac{3}{2}$

c) $V = \pi \int_1^2 \frac{9}{4x-3}\, dx$ ✓

$= \frac{9\pi}{4} \int_1^2 \frac{4}{4x-3}\, dx$

$= \frac{9\pi}{4} \left[\log_e(4x-3)\right]_1^2$ ✓

$= \frac{9\pi}{4}\left(\log_e 5 - \cancel{\log_e 1}^{\,0}\right)$

$= \frac{9\pi}{4} \log_e 5 \ u^3$ ✓

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* QUESTION 12:

a) $2^3 = x^2 - 2x$ ✓

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$\therefore x = -2$ or $x = 4$ ✓

Check $x^2 - 2x > 0$:

when $x = -2$: $(-2)^2 - 2(-2) = 8$
$> 0$

validity check not required for full marks

$x = 4$: $(4)^2 - 2(4) = 8$
$> 0$ $\therefore$ both valid

b) i) $y = x \log_e x$

Let $u = x$ $\quad v = \log_e x$

$u' = 1$ $\quad v' = \frac{1}{x}$

$y' = \log_e x \times 1 + x \times \frac{1}{x}$ ✓ Show that...

$= \log_e x + 1$

ii) $\frac{d}{dx}(x \log_e x) = \log_e x + 1$

$\therefore \int (\log_e x + 1)\, dx = x \log_e x + C_1$ ✓

$\int \log_e x\, dx + \int 1\, dx = x \log_e x + C_1$

$\therefore \int \log_e x\, dx = x \log_e x - \int 1\, dx + C_1$

$= x \log_e x - x + C_2$ ✓

c) $y = \dfrac{x}{e^x}$

i) Let $u = x$ $\quad v = e^x$

$u' = 1$ $\quad v' = e^x$

$$\frac{dy}{dx} = \frac{e^x \times 1 - x \times e^x}{(e^x)^2}$$

$$= \frac{\cancel{e^x}(1-x)}{(e^x)^{\cancel{2}}}$$

$$= \frac{1-x}{e^x}$$

Let $u = 1-x$ $\quad v = e^x$

$u' = -1$ $\quad v' = e^x$

$$\frac{d^2y}{dx^2} = \frac{e^x \times -1 - (1-x)e^x}{(e^x)^2}$$

$$= \frac{\cancel{e^x}(-1-1+x)}{(e^x)^{\cancel{2}}}$$

$$= \frac{x-2}{e^x}$$

show that...

ii) $\frac{dy}{dx} = 0$ when $x = 1$ $\quad y = \frac{1}{e}$

when $x = 1$: $\frac{d^2y}{dx^2} = \frac{1-2}{e^1}$

$= -\frac{1}{e}$

$< 0$ ∴ ∩ ∴ max t.p. occurs at $\left(1, \frac{1}{e}\right)$

iii) $\frac{d^2y}{dx^2} = 0$ when $x = 2$ $\quad y = \frac{2}{e^2}$

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $-\frac{1}{e}$ | 0 | $\frac{1}{e^3}$ |

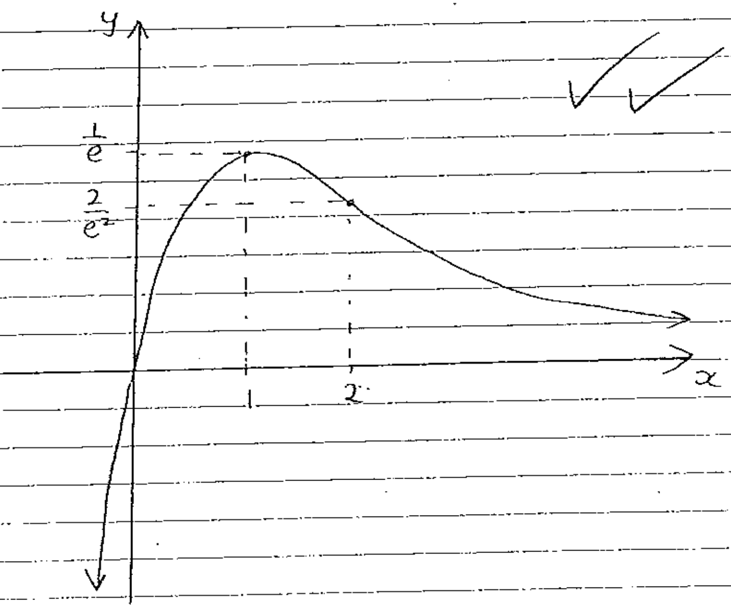
∴ change in concavity ∴ point of inflexion at $\left(2, \frac{2}{e^2}\right)$

iv) when $x = 0$, $y = 0$

max t.p. $\left(1, \frac{1}{e}\right)$ $\doteqdot 0.36\ldots$

pt of infl. $\left(2, \frac{2}{e^2}\right)$ $\doteqdot 0.27\ldots$

as $x \to \infty$, $y \to 0^+$




/13