## The Tangent to a Curve and the Derivative, and, Geometric Applications of Differentiation

1. Find a) $\lim_{x\to 5}(3x-2)$ b) $\lim_{x\to -2}\dfrac{x^2-4}{x+2}$ c) $\lim_{x\to\infty}\dfrac{3x^2+2x-9}{4x^2-11x+5}$

2. Determine whether the function $f(x)=\begin{cases}\dfrac{x^2-2x-3}{x-3} & \text{if } x\neq 3\\ 4 & \text{if } x=3\end{cases}$ is continuous at $x=3$, giving reasons.

3. From first principles, find the derivative of the function $f(x)=x^2-3x$

4. Differentiate:
   a) $x^5-x^3+7x+2$ b) $(2x-5)^3$ c) $\sqrt{9-x}$ d) $x^2\sqrt{x}-\dfrac{5}{x}$ e) $\dfrac{2x+3}{x-5}$ f) $2x(x+4)^8$

5. If $y=x^3+3x^2$, show that $y-\dfrac{dy}{dx}+\dfrac{d^2y}{dx^2}=x^3+6$

6. Find the equation of the normal to the curve $y=(3x-2)^2$, at the point where $x=0$

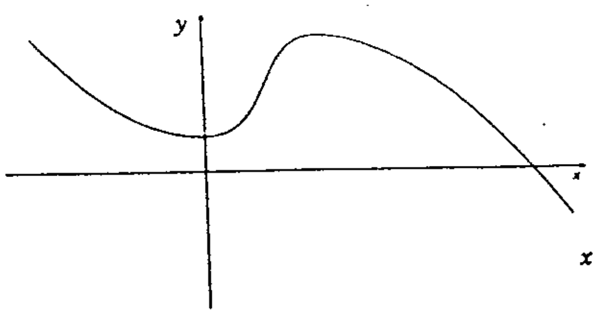
7. Show that the curve $y=\dfrac{x}{1+x^2}$ has two turning points. Determine the nature of these points and hence sketch the curve. (hint: does the curve have any asymptote(s)?)

8. Consider the curve given by $y=\dfrac{1}{4}x^4-x^3$

   a) Find any turning points and determine their nature.
   b) Find any points of inflection
   c) Sketch the curve for $-1.5\le x\le 4.5$, indicating where the curve crosses the $x$-axis
   d) For what values of $x$ is the curve concave down?

9. A closed container made up of thin metal is in the shape of a cylinder of height $h$ cm and hemispherical cap of radius $r$ cm. Find the minimum area $S$ cm² of the metal required if the container is to hold a volume of 100 cm³. (NB. For a sphere SA $=4\pi r^2$, V $=\dfrac{4}{3}\pi r^3$)

10. The diagram shows the graph of a certain function $f(x)$.




(i) Copy this graph.
(ii) On the *same* set of axes, draw a sketch of the derivative $f'(x)$ of the function.

11. Given $f''(x) = 2x-1$, and $f'(2) = -1$, $f(0) = 2$ find the equation $f(x)$.

1 a) $\lim_{x\to 5} (3x-2)$
$= 15-2$
$= 13$

b) $\lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$
$= -4$

c) $\lim_{x\to\infty} \dfrac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{9}{x^2}}{\frac{4x^2}{x^2}-\frac{11x}{x^2}+\frac{5}{x^2}}$

$= \lim_{x\to\infty} \dfrac{3+\frac{2}{x}-\frac{9}{x^2}}{4-\frac{11}{x}+\frac{5}{x^2}}$

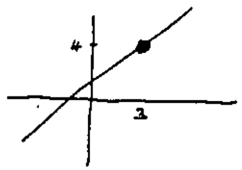
$= \frac{3}{4}$ since as $x\to\infty$ $\frac{2}{x}, \frac{9}{x^2}, \frac{11}{x}, \frac{5}{x^2} \to 0$

2. $f(x) = \begin{cases} \frac{(x-3)(x+1)}{x-3} & x \neq 3 \\ 4 & x = 3 \end{cases}$

$f(x) = \begin{cases} x+1 & x \neq 3 \\ 4 & x = 3 \end{cases}$

$f(3) = 4$
$\lim_{x\to 3^-}(x+1) = 4$
$\therefore$ fn is continuous at $x=3$
since $f(3) = \lim_{x\to 3} f(x)$

diagramatically:




3. $f(x) = x^2-3x$
$f(x+h) = (x+h)^2-3(x+h)$
$= x^2+2xh+h^2-3x-3h$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(2x+h-3)}{\cancel{h}}$

$= 2x-3$

4. a) $y = x^5-x^3+7x+2$
$\frac{dy}{dx} = 5x^4-3x^2+7$

b) $y = (2x-5)^3$
$\frac{dy}{dx} = 3(2x-5)^2(2)$
$= 6(2x-5)^2$

c) $y = (9-x)^{\frac{1}{2}}$
$\frac{dy}{dx} = \frac{1}{2}(9-x)^{-\frac{1}{2}}(-1)$
$= \frac{-1}{2\sqrt{9-x}}$

d) $y = x^{\frac{5}{2}}-5x^{-1}$
$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}+5x^{-2}$
$= \frac{5x\sqrt{x}}{2}+\frac{5}{x^2}$

4 e) $y = \frac{2x+3}{x-5}$ $\quad u = 2x+3$, $u' = 2$, $v = x-5$, $v' = 1$

$\frac{dy}{dx} = \frac{vu'-uv'}{v^2}$

$= \frac{(x-5)2-(2x+3)1}{(x-5)^2}$

$= \frac{2x-10-2x-3}{(x-5)^2}$

$= \frac{-13}{(x-5)^2}$

f) $y = 2x(x+4)^8$ $\quad u = 2x$, $u' = 2$, $v = (x+4)^8$, $v' = 8(x+4)^7$

$\frac{dy}{dx} = vu'+uv'$

$= (x+4)^8 2 + 2x.8(x+4)^7$

$= 2(x+4)^7[(x+4)+8x]$

$= 2(x+4)^7(9x+4)$

5. $y = x^3+3x^2$
$\frac{dy}{dx} = 3x^2+6x$
$\frac{d^2y}{dx^2} = 6x+6$

$y - \frac{dy}{dx} + \frac{d^2y}{dx^2}$
$= x^3+\cancel{3x^2}-\cancel{3x^2}-\cancel{6x}+\cancel{6x}+6$
$= x^3+6$
$=$ RHS.

6. $y = (3x-2)^2$
$\frac{dy}{dx} = 2(3x-2)^1(3)$
$= 6(3x-2)$
at $x=0$
$\frac{dy}{dx} = 6(0-2) = -12$
$\therefore m$ of normal $= \frac{1}{12}$

at $x=0$, $y = (0-2)^2 = 4$
$(0,4)$ $m = \frac{1}{12}$
$y-y_1 = m(x-x_1)$
$y-4 = \frac{1}{12}(x-0)$
$y = \frac{x}{12}+4$

7. $y = \frac{x}{1+x^2}$

NB when $x>0$, $y>0$
when $x<0$, $y<0$
as $x\to\infty$ $y\to 0^+$
$x\to-\infty$ $y\to 0^-$

$\frac{dy}{dx} = \frac{(1+x^2).1-x(2x)}{(1+x^2)^2}$
$= \frac{1+x^2-2x^2}{(1+x^2)^2}$
$= \frac{1-x^2}{(1+x^2)^2}$

$\frac{dy}{dx} = 0$ for stat. pts.
ie $1-x^2 = 0$
$\therefore x = \pm 1$
$x=1$, $y = \frac{1}{2}$
$x=-1$, $y = -\frac{1}{2}$ PTO.

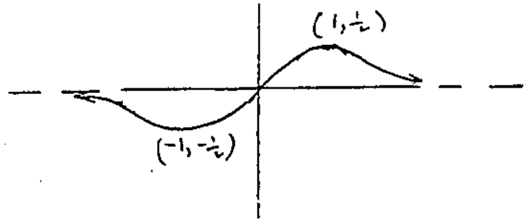
11. $f''(x) = 2x-1$
$f'(x) = x^2-x+c_1$
$\therefore c_1 = -3$
$f'(x) = x^2-x-3$
$f(0) = c_2 = 2$

Nature:

| $x$ | $-1^-$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $\frac{dy}{dx}$ | $-$ | $0$ | $+$ | $0$ | $-$ |

$\therefore$ min tp at $(-1, \frac{-1}{2})$

max tp at $(1, \frac{1}{2})$




<u>NB</u> fn is odd

$f(x) = -f(-x)$

<u>and</u> $y=0$ ($x$-axis) is an asymptote

8. $y = \frac{1}{4}x^4 - x^3$

$\frac{dy}{dx} = x^3 - 3x^2$

$\frac{d^2y}{dx^2} = 3x^2 - 6x$

For stat pts $\frac{dy}{dx} = 0$

$x^3 - 3x^2 = 0$

$x^2(x-3) = 0$

$x = 0, 3$

Nature:

at $x=0$, $\frac{d^2y}{dx^2} = 0$ $\therefore$ possibly a pt of infl. $\Rightarrow$

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | +cu | 0 | cd |

change in conc.

$\therefore$ at $x=0$ hor. pt. of infl.

at $x = 3$

$\frac{d^2y}{dx^2} > 0$ cu

$\therefore$ min tp at $x = 3$

$(0,0)$ hor. pt. of infl.

$(3, -6\frac{3}{4})$ min tp.

b) For pts of inflection

$\frac{d^2y}{dx^2} = 0$ + change in concavity

ie $3x^2 - 6x = 0$

$3x(x-2) = 0$

$\therefore x = 0, 2$

at $x = 2$

| $x$ | $2^-$ | $2$ | $2^+$ |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | cd | 0 | + cu |

$\therefore$ pt. of inflection at $(0,0)$ + $(2,-4)$

<u>$x$-intercepts</u>: $y=0$

$\therefore \frac{1}{4}x^4 = x^3$

$\therefore x^4 = 4x^3$

$x^4 - 4x^3 = 0$

$x^3(x-4) = 0$

$x = 0, 4 \ldots$

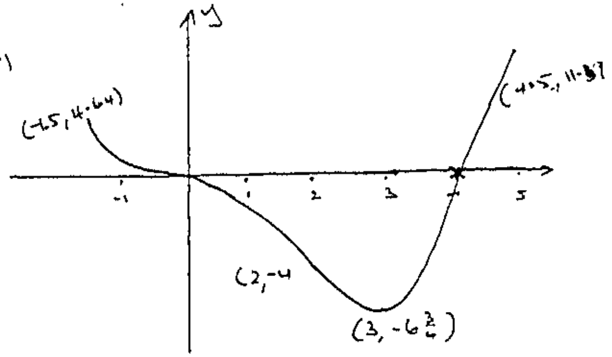
c) $x = -1.5$, $y \doteqdot 4.64$

$x = 4.5$, $y = 11.39$

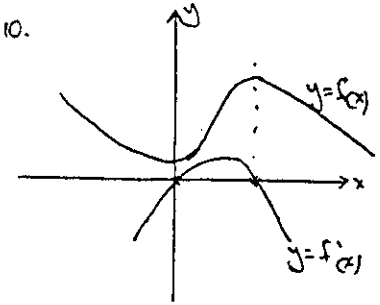
d) c.d $\Rightarrow \frac{d^2y}{dx^2} < 0$ $3x(x-2) < 0$

$0 < x < 2$

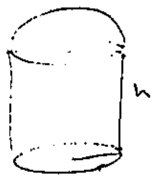
c)




10.




9.




$V = \pi r^2 h + \frac{2}{3}\pi r^3$

$\therefore 100 = \pi r^2 h + \frac{2}{3}\pi r^3$

$\therefore h = \frac{100 - \frac{2}{3}\pi r^3}{\pi r^2}$

$SA = 2\pi r h + \pi r^2 + 2\pi r^2$

$= 2\pi r h + 3\pi r^2$

$\therefore S = 2\pi r\left(\frac{100 - \frac{2}{3}\pi r^3}{\pi r^2}\right) + 3\pi r^2$

$= \frac{200}{r} - \frac{4}{3}\pi r^2 + 3\pi r^2 = 200 r^{-1} + \frac{5}{3}\pi r^2$

$\frac{dS}{dr} = -200 r^{-2} + \frac{10}{3}\pi r$

$\frac{d^2S}{dr^2} = 400 r^{-3} + \frac{10}{3}\pi$

For max/min $S$, $\frac{dS}{dr} = 0$

$\frac{-200}{r^2} + \frac{10}{3}\pi r = 0$

$\therefore \frac{200}{r^2} = \frac{10}{3}\pi r$

$600 = 10\pi r^3$

$\therefore r^3 = \frac{60}{\pi}$

$r = \sqrt[3]{\frac{60}{\pi}}$

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$\frac{d^2S}{dr^2} = \frac{400}{\frac{60}{\pi}} + \frac{10}{3}\pi > 0$

$\therefore$ cu $\therefore$ min $S$

when $r = \sqrt[3]{\frac{60}{\pi}}$

$S \doteqdot 112.233$