Student Number: /

2016 TRIAL HSC EXAMINATION

KAMBALA

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Pages 2–3

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 4–9

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 What is the value of $(9.8\times10^7)-(2.3\times10^4)$?

(A) 7.5×10^3 (B) 7.5×10^4 (C) 9.7977×10^6 (D) 9.7977×10^7

2 What is the value of $\sum_{k=1}^{4}(-1)^k k^2$?

(A) -30 (B) -10 (C) 10 (D) 30

3 Which of the following quadratic expressions is positive definite?

(A) x^2+5x+2 (B) x^2+5x+4 (C) x^2+5x+6 (D) x^2+5x+8

4 Which of the following trigonometric expressions is equivalent to $\tan\left(\frac{\pi}{2}-x\right)$?

(A) $\tan x$ (B) $\cot x$ (C) $-\tan x$ (D) $-\cot x$

5 What is the range of the function $f(x)=\sqrt{1-x^2}$?

(A) $0<y<1$ (B) $0\le y\le 1$ (C) $-1<y<1$ (D) $-1\le y\le 1$

6 α and β are the roots of the equation $x^2-8x+5=0$. What is the value of $\alpha^2+\beta^2$?

(A) 5 (B) 8 (C) 54 (D) 64

7 What is the value of $\int_{-2}^{2}|x|\,dx$?

(A) 0 (B) 4 (C) 6 (D) 8

8 What is the amplitude and period of the function $f(x) = 2 - \sin 2x$?

(A) Amplitude = 1, Period = π (B) Amplitude = 1, Period = 2π

(C) Amplitude = 2, Period = π (D) Amplitude = 2, Period = 2π

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9 Which of the following is an expression for $\frac{d}{dx}[e^{2x} \tan x]$?

(A) $2e^{2x} \tan x$ (B) $e^{2x} \sec^2 x$ (C) $2e^{2x}(1 + \tan^2 x)$ (D) $e^{2x}(1 + \tan x)^2$

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10 What is the number of real roots of the equation $x(x-2) \log_e x = 0$?

(A) 0 (B) 1 (C) 2 (D) 3

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Expand and simplify $(3 - 2\sqrt{2})^2$. **1**

(b) Solve the quadratic equation $2x^2 - 5x - 3 = 0$. **2**

(c) Differentiate each of the following with respect to x.

(i) $\sin(x - \pi)$ **1**

(ii) $x \log_e x$ **2**

(d) Find the equation of the tangent to the curve $y = \frac{x}{x-1}$ at the point (2,2) on the curve. **3**

(e) Evaluate the integral $\int_1^2 \frac{x+1}{x}\,dx$. Give your answer in simplest exact form. **3**

(f) The region bound by the curve $y = \frac{1}{2x+1}$ and the x axis between $x = 0$ and $x = 1$ is rotated about the x-axis to form a solid. Find the volume of the solid of revolution. Give your answer in exact form. **3**

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Find $\int \sec x(\cos x + \sec x)\,dx$. **2**

(b) Show that the curve $y = x^3 - 3x^2 + 6x$ has a point of inflexion at $(1,4)$. **3**

(c) A parabola has equation $8y = x^2 - 6x + 1$.

(i) Write the equation in the form $(x-h)^2 = 4a(y-k)$. **1**

(ii) Find the coordinates of the vertex and the focal length of the parabola. **2**

(iii) Draw a large, neat sketch of the parabola. Give the coordinates of the focus and the equation of the directrix of the parabola on your sketch. **2**

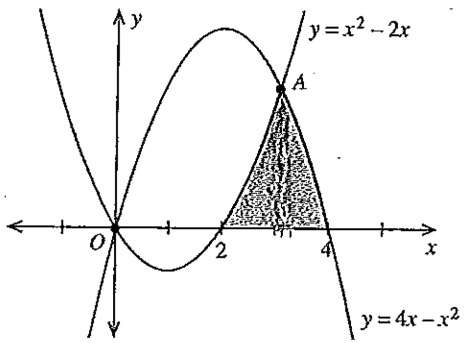
(d) Find the values of x for which the function $y = x - x^2$ is decreasing. **2**

(e) The 8th term of an arithmetic progression is 23. The 11th term is four times the 3rd term. Find the first term and the common difference of the arithmetic progression. **3**

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) The gradient function of a curve $y = f(x)$ is given by $f'(x) = \frac{x}{2} + \frac{4}{\sqrt{x}}$. The curve passes through the point $(4,5)$. Find the equation of the curve. **3**

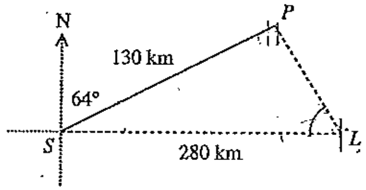
(b) The diagram below shows the parabolas $y = 4x - x^2$ and $y = x^2 - 2x$. The graphs intersect at the origin O and the point A.



(i) Find the coordinates of the point A. **1**

(ii) Find the area of the shaded region bound by the two parabolas and the x-axis. **3**

(c) A ship S is 280 km west of a lighthouse L. It travels a distance of 130 km on a bearing of 064° to a position P, as shown in the diagram below.



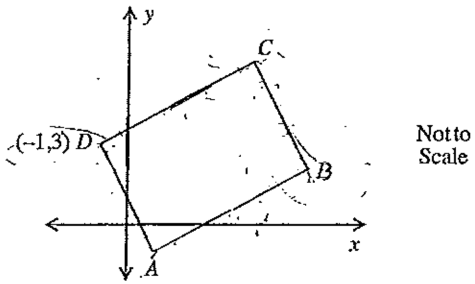
(i) Calculate the distance from the lighthouse to the ship's position at P. **2**

(ii) Find the bearing of P from the lighthouse L. **2**

(d) Use Simpson's rule with 5 function values to approximate $\int_1^9 (\log_e x)^2\,dx$. Give your answer correct to 2 significant figures. **4**

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) In the diagram below, $ABCD$ is a parallelogram. The equation of side AB is $x - 2y - 3 = 0$. The equation of side BC is $2x + y - 16 = 0$. Vertex D has coordinates $(-1,3)$.



(i) Show that $ABCD$ is a rectangle. **2**

(ii) Find the coordinates of vertex C. **3**

(b) The population of a colony of laboratory rats is modelled using the equation

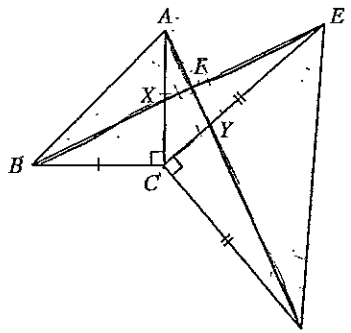
$P = P_0 e^{kt}$.

where k is a constant, t is the time (in weeks) and P_0 is the initial population of the colony.

(i) A population of 20 rats increases to 100 rats after 6 weeks. Calculate the value of k. Give your answer correct to 4 decimal places. **2**

(ii) How long will it take for the population of the colony to reach 500 rats? **2**

(c) In $\Delta ABC, AC = BC$ and $\angle BCA$ is a right angle. In $\Delta CDE, DC = EC$ and $\angle ECD$ is a right angle. DA and BE intersect at F. CA and BE intersect at X. CE and AD intersect at Y.



(i) Copy the diagram into your writing booklet.

(ii) Prove that $\Delta BCE \equiv \Delta ACD$. **3**

(iii) Show that DA is perpendicular to BE. **3**

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) Find the value of $\int_0^{\ln 3} e^{2x}\, dx$. **2**

(b) Solve the equation: $2\log_2 x - \log_2(x+4) = 1$ **3**

(c) A point P moves so that its distance from $A(3,0)$ is twice the distance from $B(0,3)$.

(i) Show that the equation of the locus of P is $x^2 + 2x + y^2 - 8y + 9 = 0$. **2**

(ii) Show that this locus is a circle and write down its centre and radius. **2**

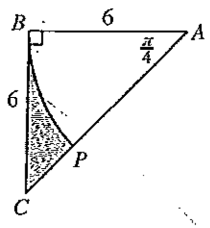
(d) (i) Solve the equation $\sin x = \cos x$ for $0 \le x \le 2\pi$. **1**

(ii) On the same diagram, sketch the graphs of the curves $y = \sin x$ and $y = \cos x$ for the domain $0 \le x \le 2\pi$. Clearly show the intercepts on the coordinate axes. **2**

(iii) Find the area of the region bounded by the curves $y = \sin x$ and $y = \cos x$ in the domain $0 \le x \le 2\pi$. Give your answer in exact form. **3**

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) In the diagram below, ΔABC is a triangle with $\angle ABC = 90°$. $AB = BC = 6$ cm and $\angle CAB = \frac{\pi}{4}$ radians. PB is an arc of a circle with centre A and radius AB.



(i) Find the exact area of sector ABP. **1**

(ii) Find the exact area of the shaded portion BPC. **1**

(b) Determine an expression for the limiting sum of the series $k, \frac{k}{e}, \frac{k}{e^2}, \ldots$ **2**

(c) A cylindrical container, closed at both ends, is to be made from thin sheet metal. The container is to have a radius of r cm and a height of h cm. Its volume is 2000π cm^3.

(i) Show that the area of sheet metal required to make the container is $\left(2\pi r^2 + \frac{4000\pi}{r}\right)$ cm^2. **2**

(ii) Find the radius required so that area of sheet metal required to make the container is minimised. **3**

(ii) Hence find the minimum area of sheet metal required to make the container. **1**

(d) Consider the equation $\frac{x}{a} + \frac{a+2}{x+1} = 2$ (with $a \neq 0$ and $x \neq -1$).

(i) Show that $x^2 + x(1-2a) + a^2 = 0$. **2**

(ii) Find the greatest integer a for which x is real and rational. **3**

End of paper

2 UNIT MATHEMATICS
2016 TRIAL HSC EXAMINATION

SECTION I

1 $(9.8 \times 10^7) - (2.3 \times 10^4) = 97\,977\,000$
$= 9.7977 \times 10^7$ — 1 D

2 $\sum_{k=1}^{4}(-1)^k k^2 = [(-1)^1 \times (1)^2] + [(-1)^2 \times (2)^2] + [(-1)^3 \times (3)^2] + [(-1)^4 \times (4)^2]$ — 2 C
$= [-1 \times 1] + [1 \times 4] + [-1 \times 9] + [1 \times 16]$
$= (-1) + 4 + (-9) + 16$
$= 10$

3 An expression is positive definite when $a > 0$ and $\Delta < 0$. — 3 D
Each expression takes the form $x^2 + 5x + k$.
$\therefore$ We need:
$b^2 - 4ac < 0$
$(5)^2 - 4(1)(k) < 0$
$25 - 4k < 0$
$-4k < -25$
$k > 6\frac{1}{4}$
$\therefore$ Only the expression $x^2 + 5x + 8$ is positive definite.

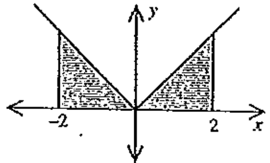
4 $\tan(\frac{\pi}{2} - x) = \tan(90 - x)$ — 4 B
$= \cot x$

5 For the domain: — 5 B
$1 - x^2 \geq 0$
$(1 + x)(1 - x) \geq 0$
$-1 \leq x \leq 1$
$f(-1) = \sqrt{1 - (-1)^2}$
$= \sqrt{1 - 1}$
$= 0$
$f(0) = \sqrt{1 - (0)^2}$
$= \sqrt{1 - 0}$
$= 1$
$f(1) = \sqrt{1 - (1)^2}$
$= \sqrt{1 - 1}$
$= 0$
$\therefore$ Range: $\{y: 0 \leq y \leq 1\}$

6 $\alpha + \beta = \frac{-b}{a}$ — 6 C
$= \frac{-(-8)}{1}$
$= 8$

$\alpha\beta = \frac{c}{a}$
$= \frac{5}{1}$
$= 5$
$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (8)^2 - 2(5)$
$= 64 - 10$
$= 54$

7 The expression $\int_{-2}^{2} |x|\, dx$ represents the area under the graph $y = |x|$ from $x = -2$ to $x = 2$. — 7 B



Area $= 2 \times \frac{1}{2}bh$
$= 2 \times \frac{1}{2} \times 2 \times 2$
$= 4$
$\therefore \int_{-2}^{2} |x|\, dx = 4$

8 Amplitude $= 1$ — 8 A
Period $= \frac{2\pi}{2}$
$= \pi$

9 $\frac{d}{dx}[e^{2x} \tan x] = e^{2x} \times \sec^2 x + \tan x \times 2e^{2x}$ — 9 D
$= e^{2x}(\sec^2 x + 2 \tan x)$
$= e^{2x}(1 + \tan^2 x + 2 \tan x)$
$= e^{2x}(\tan^2 x + 2 \tan x + 1)$
$= e^{2x}(\tan x + 1)^2$

10 Make each factor equal 0. — 10 C
Therefore:
$x = 0$
or:
$x - 2 = 0$
$x = 2$
or:
$\ln x = 0$
$x = 1$

However, $\ln x$ is only defined for $x > 0$.
$\therefore x = 0$ is not a valid solution.
$\therefore$ There are only 2 real roots of the equation.

SECTION II

QUESTION 11

(a) $(3-2\sqrt{2})^2 = 3^2 - 2\times 3\times 2\sqrt{2} + (2\sqrt{2})^2$
$= 9 - 12\sqrt{2} + 4\times 2$
$= 17 - 12\sqrt{2}$

(b) $2x^2 - 5x - 3 = 0$
$(2x+1)(x-3) = 0$
Therefore:
$2x+1=0$
$2x=-1$
$x=-\frac{1}{2}$
or:
$x-3=0$
$x=3$
$\therefore$ Solution is $x=-\frac{1}{2}$ or 3

(c) (i) $y=\sin(x-\pi)$
$\frac{dy}{dx}=\cos(x-\pi)$
(ii) $y = x\ln x$
$\frac{dy}{dx} = x.\frac{1}{x} + \ln x.1$
$= 1+\ln x$

(d) $y=\frac{x}{x-1}$
$\frac{dy}{dx} = \frac{(x-1).1-x.1}{(x-1)^2}$
$=\frac{x-1-x}{(x-1)^2}$
$=-\frac{1}{(x-1)^2}$
At (2,2), $\frac{dy}{dx} = -\frac{1}{(2-1)^2}$
$=-\frac{1}{1^2}$
$=-1$
$\therefore$ Gradient of tangent to curve at $(2,2)=-1$
$\therefore$ Equation of tangent to curve is:
$y-y_1=m(x-x_1)$
$y-2=-1(x-2)$
$y-2=-x+2$
$x+y-4=0$

(e) $\int_1^2 \frac{x+1}{x}\,dx = \int_1^2 \frac{x}{x}+\frac{1}{x}\,dx$
$=\int_1^2 1+\frac{1}{x}\,dx$
$=[x+\ln x]_1^2$
$=[2+\ln 2]-[1+\ln 1]$
$=2+\ln 2-1-\ln 1$
$=1+\ln 2$

(f) Equation of curve is:
$y=\frac{1}{2x+1}$
$=(2x+1)^{-1}$
$y^2=(2x+1)^{-2}$
Volume $=\pi\int_0^1 (2x+1)^{-2}\,dx$
$=\pi\left[-\frac{1}{2}(2x+1)^{-1}\right]_0^1$
$=\pi[-\frac{1}{2}(2(1)+1)^{-1}]-\pi[-\frac{1}{2}(2(0)+1)^{-1}]$
$=\pi[-\frac{1}{2}(2+1)^{-1}]-\pi[-\frac{1}{2}(0+1)^{-1}]$
$=\pi[-\frac{1}{2}(3)^{-1}]-\pi[-\frac{1}{2}(1)^{-1}]$
$=\pi[-\frac{1}{2}\times\frac{1}{3}]-\pi[-\frac{1}{2}\times 1]$
$=-\frac{\pi}{6}+\frac{\pi}{2}$
$=\frac{\pi}{3}$ cubic units

QUESTION 12

(a) $\int \sec x(\cos x+\sec x)\,dx = \int \sec x\cos x+\sec^2 x\,dx$
$=\int \frac{1}{\cos x}\times\frac{\cos x}{1}+\sec^2 x\,dx$
$=\int 1+\sec^2 x\,dx$
$=x+\tan x+C$

(b) $y=x^3-3x^2+6x$
$\frac{dy}{dx}=3x^2-6x+6$
$\frac{d^2y}{dx^2}=6x-6$
For points of inflection, $\frac{d^2y}{dx^2}=0$ and changes sign on either side of the point.
$6x-6=0$
$6x=6$
$x=1$
When $x=1, y=4$

When $x = 0.9$, $\frac{d^2y}{dx^2} = -0.6 < 0$

When $x = 1.1$, $\frac{d^2y}{dx^2} = 0.6 > 0$

$\therefore$ Point of inflection at (1,4)

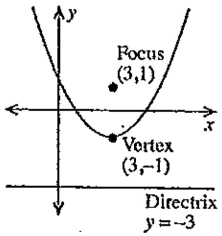
(c) (i) Equation of the parabola is:

$$8y = x^2 - 6x + 1$$
$$x^2 - 6x = 8y - 1$$
$$x^2 - 6x + 9 = 8y - 1 + 9$$
$$(x-3)^2 = 8y + 8$$
$$(x-3)^2 = 8(y+1)$$
$$(x-3)^2 = 4.2.(y+1)$$

(ii) Vertex is (3,–1)

Focal length is 2 units

(iii)



Focus is (3,1)

Equation of directrix is $y = -3$

(d) $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

Curve is decreasing when $\frac{dy}{dx} < 0$

$$1 - 2x < 0$$
$$-2x < -1$$
$$x > \tfrac{1}{2}$$

$\therefore$ Curve is decreasing when $x > \frac{1}{2}$

(e) We have:

$$T_8 = 23$$
$$a + (8-1)d = 23$$
$$a + 7d = 23$$

and:

$$T_{11} = 4T_3$$
$$a + (11-1)d = 4[a + (3-1)d]$$
$$a + 10d = 4[a + 2d]$$
$$a + 10d = 4a + 8d$$
$$3a - 2d = 0$$

Solving simultaneously:

$$\begin{cases} a + 7d = 23 \\ 3a - 2d = 0 \end{cases}$$
$$\begin{cases} 2a + 14d = 46 \\ 21a - 14d = 0 \end{cases}$$
$$23a = 46$$
$$a = 2$$

Substituting:

$$2 + 7d = 23$$
$$7d = 21$$
$$d = 3$$

$\therefore$ The first term of the arithmetic progression is 2 and the common difference is 3.
The sequence is 2, 5, 8, ...

QUESTION 13

(a) $f'(x) = \frac{x}{2} + \frac{4}{\sqrt{x}}$

$$= \tfrac{1}{2}x + 4x^{-\frac{1}{2}}$$
$$f(x) = \tfrac{1}{4}x^2 + 8x^{\frac{1}{2}} + C$$
$$= \frac{x^2}{4} + 8\sqrt{x} + C$$

The curve passes through (4,5).

$$\frac{(4)^2}{4} + 8\sqrt{4} + C = 5$$
$$\frac{16}{4} + 8 \times 2 + C = 5$$
$$4 + 16 + C = 5$$
$$20 + C = 5$$
$$C = -15$$
$$\therefore f(x) = \frac{x^2}{4} + 2\sqrt{x} - 15$$

(b) (i) Solving for coordinates of A:

$$\begin{cases} y = x^2 - 2x \\ y = 4x - x^2 \end{cases}$$
$$x^2 - 2x = 4x - x^2$$
$$2x^2 - 6x = 0$$
$$2x(x-3) = 0$$
$$x = 0 \text{ and } x = 3$$

When $x = 0, y = 0$

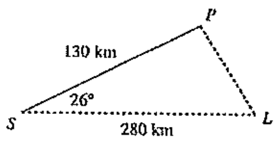
When $x = 3, y = 3$

$\therefore A = (3,3)$

(b) (ii) Area $= \int_2^3 x^2 - 2x\,dx + \int_3^4 4x - x^2\,dx$

$$= \left[\tfrac{1}{3}x^3 - x^2\right]_2^3 + \left[2x^2 - \tfrac{1}{3}x^3\right]_3^4$$
$$= [\tfrac{1}{3}(3)^3 - (3)^2] - [\tfrac{1}{3}(2)^3 - (2)^2] + [2(4)^2 - \tfrac{1}{3}(4)^3] - [2(3)^2 - \tfrac{1}{3}(3)^3]$$
$$= [9 - 9] - [2\tfrac{2}{3} - 4] + [32 - 21\tfrac{1}{3}] - [18 - 9]$$
$$= 0 - (-1\tfrac{1}{3}) + 10\tfrac{2}{3} - 9$$
$$= 3 \text{ square units}$$

(c) (i)

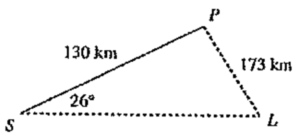


Using the cosine rule:

$$PL^2 = 130^2 + 280^2 - 2 \times 130 \times 280 \times \cos 26°$$
$$= 29867.79343$$
$$PL = 172.8230119$$
$$\approx 173 \text{ km}$$

∴ The ship is approximately 173 km from the lighthouse.

(ii)



Using the sine rule:

$$\frac{130}{\sin\angle PLS} = \frac{173}{\sin 26°}$$
$$173\sin\angle PLS = 130\sin 26°$$
$$\sin\angle PLS = \frac{130\sin 26°}{173}$$
$$= 0.3294118444$$
$$\angle PLS = 19.23308874$$
$$\approx 19°$$

∴ Bearing of P from L is 289°.

(d) Using Simpson's rule:

x	$f(x)$	W	P
1	0.0000	1	0.0000
3	1.2069	4	4.8278
5	2.5903	2	5.1806
7	3.7866	4	15.1463
9	4.8278	1	4.8278
			29.9825

Therefore:

$$\int_1^9 (\ln x)^2\,dx \approx \tfrac{1}{3} \times h \times \text{sum}$$
$$= \tfrac{1}{3} \times 2 \times 29.9825$$
$$= 19.9883$$
$$\approx 20$$

QUESTION 14

(a) (i) $ABCD$ is a parallelogram (given)

$AB = DC$ (opp sides of parallelogram)

$AD = BC$ (opp sides of parallelogram)

Gradient of $AB = \dfrac{-a}{b} = \dfrac{-1}{-2} = \tfrac{1}{2}$

Gradient of $BC = \dfrac{-a}{b} = \dfrac{-2}{1} = -2$

$m_{AB} \times m_{BC} = \tfrac{1}{2} \times -2 = -1$

∴ AB is perpendicular to BC

$AB \parallel DC$ (opp sides of parallelogram)

∴ BC is perpendicular to CD

$AD \parallel BC$ (opp sides of parallelogram)

∴ CD is perpendicular to AD

∴ $ABCD$ is a rectangle (all angles are 90°)

(ii) Equation of BC is $2x + y - 16 = 0$

Gradient of $AB = \tfrac{1}{2}$

Now CD is parallel to AB and passes through $(-1,3)$

Equation of CD is:

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \tfrac{1}{2}(x + 1)$$
$$y - 3 = \tfrac{1}{2}x + \tfrac{1}{2}$$
$$\tfrac{1}{2}x - y + \tfrac{7}{2} = 0$$
$$x - 2y + 7 = 0$$

Now, solving for coordinates of C:

$$\begin{cases} 2x+y-16=0 \\ x-2y+7=0 \end{cases}$$

$$\begin{cases} 2x+y-16=0 \\ 2x-4y+14=0 \end{cases}$$

$$5y-30=0$$
$$5y=30$$
$$y=6$$

Substituting:

$$2x+6-16=0$$
$$2x-10=0$$
$$2x=10$$
$$x=5$$

$\therefore C \equiv (5,6)$

(b) (i) $P=P_0e^{kt}$

When $t=0$, let $P=20$.

$$20=P_0\times e^{k(0)}$$
$$20=P_0\times e^0$$
$$P_0=20$$

$\therefore P=20e^{kt}$

When $t=6$, $P=100$.

$$100=20e^{k(6)}$$
$$5=e^{6k}$$
$$\ln 5=\ln e^{6k}$$
$$6k=\ln 5$$
$$k=\frac{\ln 5}{6}$$
$$=0.2682$$

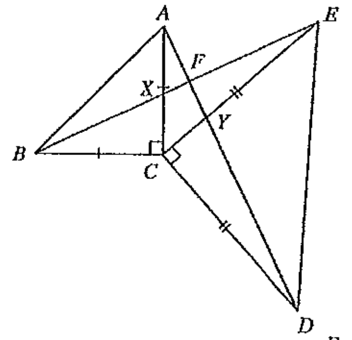
$\therefore P=20e^{0.2682t}$

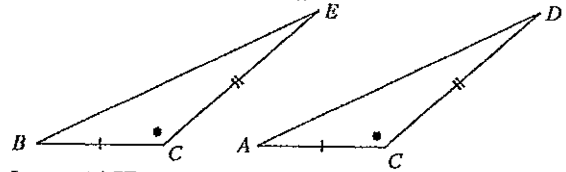
(ii) When $P=500$,

$$500=20e^{0.2682t}$$
$$25=e^{0.2682t}$$
$$\ln 25=\ln e^{0.2682t}$$
$$0.2682t=\ln 25$$
$$t=\frac{\ln 25}{0.2682}$$
$$=12.00177414$$

$\therefore$ The population of the colony will reach 500 after approximately 12 weeks.

(c) (i)





(ii) Let $x=\angle ACE$

$\angle BCE=90+x$	(adjacent angles)
$\angle ACD=x+90$	(adjacent angles)
$\therefore \angle BCE=\angle ACD$	
$BC=AC$	(given)
$CE=CD$	(given)
$\therefore \Delta BCE\equiv\Delta ACD$	(SAS)

(iii) ΔECD is an isosceles right angled triangle

$\angle CDE=\angle CED=45°$	(equal angles opp equal sides)
In ΔFED:	
$\angle FED=\angle FEY+\angle YED$	(adjacent angles)
$=\angle FEY+45°$	
$=\angle BEC+45°$	
$\angle BEC=\angle FED-45°$	
and:	
$\angle CDE=\angle CDY+\angle FDE$	(adjacent angles)
$45°=\angle CDY+\angle FDE$	
$\angle CDY=45°-\angle FDE$	
$\angle CDA=45°-\angle FDE$	
Now:	
$\angle BEC=\angle ACD$	(corr angles in cong triangles)
$\angle FED-45°=45°-\angle FDE$	
$\angle FED=90°-\angle FDE$	
$\angle FED+\angle FDE=90°$	
Therefore:	
$\angle FED+\angle EDF+\angle DFE=180°$	(angle sum of Δ)
$90°+\angle DFE=180°$	
$\angle DFE=90°$	

$\therefore DA$ is perpendicular to BE

QUESTION 15

(a) $\int_0^{\ln 3} e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^{\ln 3}$

$= \frac{1}{2}e^{2(\ln 3)} - \frac{1}{2}e^{2(0)}$

$= \frac{1}{2}e^{\ln 3^2} - \frac{1}{2}e^0$

$= \frac{1}{2}e^{\ln 9} - \frac{1}{2}$

$= \frac{1}{2}(9) - \frac{1}{2}$

$= 4$

(b)

$2\log_2 x - \log_2(x+4) = 1$

$\log_2 x^2 - \log_2(x+4) = 1$

$\log_2\left(\frac{x^2}{x+4}\right) = 1$

Using the definition of the logarithm:

$2^1 = \frac{x^2}{x+4}$

$2(x+4) = x^2$

$2x + 8 = x^2$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$\therefore x = 4$ or $x = -2$

But $\log_2 x$ is only defined for $x > 0$.

$\therefore$ Solution is $x = 4$

(c) (i) Let $P(X,Y)$ be one position of the variable point.
Now:

$PA = 2PB$

$PA^2 = (2PB)^2$

$PA^2 = 4PB^2$

$(X-3)^2 + (Y-0)^2 = 4[(X-0)^2 + (Y-3)^2]$

$X^2 - 6X + 9 + Y^2 = 4[X^2 + Y^2 - 6Y + 9]$

$X^2 - 6X + 9 + Y^2 = 4X^2 + 4Y^2 - 24Y + 36$

$3X^2 + 6X + 3Y^2 - 24Y + 27 = 0$

$3(X^2 + 2X + Y^2 - 8Y + 9) = 0$

$X^2 + 2X + Y^2 - 8Y + 9 = 0$

$\therefore$ The equation of the locus of P is $x^2 + 2x + y^2 - 8y + 9 = 0$.

(ii) Completing the square:

$x^2 + 2x + y^2 - 8y + 9 = 0$

$x^2 + 2x + y^2 - 8y = -9$

$x^2 + 2x + 1 + y^2 - 8y + 16 = -9 + 1 + 16$

$(x+1)^2 + (y-4)^2 = 8$

$\therefore$ The locus of P is a circle with centre $(-1,4)$ and radius $\sqrt{8}$ units.

(d) (i)

$\sin x = \cos x$

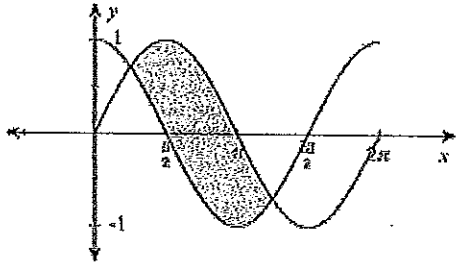
$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = 45°$ or $225°$

$= \frac{\pi}{4}$ or $\frac{5\pi}{4}$

(ii)



(iii) Solving for points of intersection:

$\begin{cases} y = \sin x \\ y = \cos x \end{cases}$

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

Now:

Area $= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x\,dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos x\,dx$

$= \left[-\cos x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$= \left[[-\cos\frac{5\pi}{4}] - [-\cos\frac{\pi}{4}]\right] - \left[[\sin\frac{5\pi}{4}] - [\sin\frac{\pi}{4}]\right]$

$= \left[[-(-\frac{1}{\sqrt{2}})] - [-(\frac{1}{\sqrt{2}})]\right] - \left[[-\frac{1}{\sqrt{2}}] - [\frac{1}{\sqrt{2}}]\right]$

$= \left[[\frac{1}{\sqrt{2}}] - [-\frac{1}{\sqrt{2}}]\right] - \left[[-\frac{1}{\sqrt{2}}] - [\frac{1}{\sqrt{2}}]\right]$

$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$

$= \left[\frac{2}{\sqrt{2}}\right] - \left[-\frac{2}{\sqrt{2}}\right]$

$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$

$= \frac{4}{\sqrt{2}}$

$= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{4\sqrt{2}}{2}$

$= 2\sqrt{2}$ square units

QUESTION 16

(a) (i) Area of sector $ABP = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 6^2 \times \frac{\pi}{4}$

$= \frac{9\pi}{2}$ square units

(ii) Area $\Delta ABC = \frac{1}{2}bh$

$= \frac{1}{2} \times 6 \times 6$

$= 18$ square units

Area $ABP = 18 - \frac{9\pi}{2}$

$= \frac{36-9\pi}{2}$ square units

(b) First term $= k$

Common ratio $= \frac{k}{e} \div k$

$= \frac{k}{e} \times \frac{1}{k}$

$= \frac{1}{e}$

Limiting sum $= \frac{a}{1-r}$

$= \dfrac{k}{1-\frac{1}{e}}$

$= k \div \left(1-\frac{1}{e}\right)$

$= k \div \frac{e-1}{e}$

$= k \times \frac{e}{e-1}$

$= \frac{ke}{e-1}$

(c) (i) The volume of the cylinder is 2000π cm^3.

Therefore:

$\pi r^2 h = 2000\pi$

$r^2 h = 2000$

$h = \frac{2000}{r^2}$

Now let S be the surface area of sheet metal required to make the container.

$S = 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r\left(\frac{2000}{r^2}\right)$

$= 2\pi r^2 + \left(\frac{2\pi r}{1} \times \frac{2000}{r^2}\right)$

$= 2\pi r^2 + \frac{4000\pi r}{r^2}$

$= 2\pi r^2 + \frac{4000\pi}{r}$

(ii) $S = 2\pi r^2 + 4000\pi r^{-1}$

$\frac{dS}{dr} = 4\pi r - 4000\pi r^{-2}$

$\frac{d^2S}{dr^2} = 4\pi + 8000\pi r^{-3}$

For minimum area, we need $\frac{dS}{dr} = 0$ and $\frac{d^2S}{dr^2} > 0$.

$4\pi r - 4000\pi r^{-2} = 0$

$4\pi r - \frac{4000\pi}{r^2} = 0$

$4\pi r = \frac{4000\pi}{r^2}$

$4\pi r^3 = 4000\pi$

$r^3 = 1000$

$r = 10$

When $r = 10$,

$\frac{d^2S}{dr^2} = 4\pi + \frac{8000\pi}{(10)^3}$

$= 4\pi + \frac{8000\pi}{1000}$

$= 4\pi + 8\pi$

$= 12\pi$

> 0

$\therefore$ The minimum area of sheet metal required occurs when the radius is 10 cm.

(iii) When $r = 10$,

$S = 2\pi(10)^2 + \frac{4000\pi}{10}$

$= 200\pi + 400\pi$

$= 600\pi$

$\therefore$ The minimum area of sheet metal required is 600π cm^2.

(d) (i)

$\frac{x}{a} + \frac{a+2}{x+1} = 2$

$\frac{x(x+1)}{a(x+1)} + \frac{a(a+2)}{a(x+1)} = 2$

$\frac{x(x+1)+a(a+2)}{a(x+1)} = 2$

$x(x+1) + a(a+2) = 2a(x+1)$

$x^2 + x + a^2 + 2a = 2ax + 2a$

$x^2 + x + a^2 = 2ax$

$x^2 + x - 2ax + a^2 = 0$

$x^2 + (1-2a)x + a^2 = 0$

(ii) For the discriminant,

$$\Delta = b^2 - 4ac$$
$$= (1-2a)^2 - 4(1)(a^2)$$
$$= 1 - 4a + 4a^2 - 4a^2$$
$$= 1 - 4a$$

For real and rational roots, $\Delta \geq 0$ and Δ is a perfect square.
Therefore:

$$1 - 4a \geq 0$$
$$-4a \geq -1$$
$$a \leq \tfrac{1}{4}$$

Since a is an integer and $a \neq 0$, we have $a = -1, -2, -3, \ldots$
We also need Δ to be a perfect square.
When $a = -1$, $\Delta = 1 - 4(-1)$
$= 1 + 4$
$= 5$, which is not a perfect square.
When $a = -2$, $\Delta = 1 - 4(-2)$
$= 1 + 8$
$= 9$, which is a perfect square.
$\therefore$ The greatest value for a is -2.

(ii) For the discriminant,

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (1-2a)^2 - 4(1)(a^2) \\ &= 1 - 4a + 4a^2 - 4a^2 \\ &= 1 - 4a\end{aligned}$$

For real and rational roots, $\Delta \geq 0$ and Δ is a perfect square.
Therefore:

$$\begin{aligned}1 - 4a &\geq 0 \\ -4a &\geq -1 \\ a &\leq \tfrac{1}{4}\end{aligned}$$

Since a is an integer and $a \neq 0$, we have $a = -1, -2, -3, \ldots$
We also need Δ to be a perfect square.

When $a = -1$, $\Delta = 1 - 4(-1)$
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$= 1 + 8$
$= 9$, which is a perfect square.

$\therefore$ The greatest value for a is -2.