Raising Indices to Indices

We use what we know about multiplying terms with indices, to find a rule for raising indices to indices.
Consider for example:

$$(2^4)^2 = 2^4 \times 2^4 = 2^8 = 2^{4\times 2}$$

Notice that the base remains the same and we find the product of the indices. In general, apply the formula

$$(a^m)^n = a^{mn}$$

Same base — Multiply indices

Raising indices to indices

ⓐ $(3^5)^2 = 3^{5\times 2} = 3^{10}$

ⓑ $(4^3)^x = 4^{3\times x} = 4^{3x}$

More Index Laws

If a product or fraction is raised to an index, then the index applies to each term.

$$(ab)^m = a^m b^m \quad \text{and} \quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Brackets with indices

ⓐ $(3x)^3 = 3^3x^3$
$= 27x^3$

ⓑ $(4pq^2)^4 = 4^4p^4(q^2)^4$
$= 256p^4q^{2\times 4}$
$= 256p^4q^8$

ⓒ $\left(\frac{x}{2}\right)^3 = \frac{x^3}{2^3}$
$= \frac{x^3}{8}$

ⓓ $\left(\frac{3a^3b}{5}\right)^2 = \frac{3^2(a^3)^2b^2}{5^2}$
$= \frac{9a^6b^2}{25}$

The Zero Index

$$a^0 = a^{m-m} = \frac{a^m}{a^m} = 1$$

This means $a^0 = 1$ when $a \neq 0$

Anything raised to an index of 0 is 1.

The zero index

ⓐ $4^0 = 1 = 5^0 = 1 = 100000^0 = 1 = (-5)^0 = 1 = \left(\frac{2}{3}\right)^0 = 1 = (a+b)^0 = 1 = x^0 = 1 = (a^3b^2)^0 = 1$

ⓑ $6^0 + 2(6^0) = 1 + 2(1) = 3$

ⓒ $(ab)^0 \times 7 = 1 \times 7 = 7$

Fractional Indices

Let's try figure out what to do when the indices are fractions, such as $16^{\frac{1}{4}}$ or $p^{\frac{1}{7}}$.

For example, consider $5^{\frac{1}{2}}$:

1. Using the index law for multiplication we can say $(5^{\frac{1}{2}})^2 = 5^1 = 5$
2. Find the square root of both sides to obtain $\sqrt{(5^{\frac{1}{2}})^2} = \sqrt{5}$
3. Simplify by cancelling the index of 2 with the square root $5^{\frac{1}{2}} = \sqrt{5}$

For any a, we can say $a^{\frac{1}{2}}$ is the **square root** of the number a. In the same way $a^{\frac{1}{3}}$ is the **cube root** of a.
Basically, for any n, $a^{\frac{1}{n}}$ is the n^{th} **root** of a. Always use the formula

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Fractional indices

ⓐ $36^{\frac{1}{2}} = \sqrt{36} = 6$

ⓑ $(8x^{12})^{\frac{1}{3}} = 8^{\frac{1}{3}} \times x^{12\times\frac{1}{3}} = 2x^4$

Negative Indices

Let's see if we can use what we know to figure out how to use negative indices such as $3^{-1}, (-2)^{-10}$ or x^{-5}.

For example:

Negative indices

$5^{-2} = 5^{0-2}$ ← Since $0 - 2 = -2$

$5^{0-2} = \frac{5^0}{5^2}$ ← Using the division of indices law

$\frac{5^0}{5^2} = \frac{1}{5^2}$ ← According to the zero index law

$\therefore 5^{-2} = \frac{1}{5^2}$

In general, for negative indices we use the formula:

$$a^{-n} = \frac{1}{a^n}$$

Negative indices examples

a) $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

b) $\left(\frac{t^3}{3}\right)^{-1} = 1/\frac{t^3}{3} = \frac{3}{t^3}$

c) $4x^{-2} = \frac{4}{x^2}$

d) $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2} = \frac{25}{9}$

e) $\frac{1}{2a} = \frac{1}{2}a^{-1}$

f) $(-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4}$ ← In this example, the minus (−) is included in the index.

g) $-2^{-2} = -\frac{1}{(2)^2} = -\frac{1}{4}$ ← In this example, it is NOT included in the index.

More Fractional Indices

Up until now we've only worked with fractional indices with numerator 1 like $25^{\frac{1}{2}}$ or $b^{\frac{1}{5}}$.

As all mathematicians know, these are not the only type of fractions. It is important to learn how to use fractional indices whose numerators are not 1 for example: $5^{\frac{3}{2}}$

Using the raising an index to an index law, we find

$$4^{\frac{3}{5}} = (4^3)^{\frac{1}{5}} = \sqrt[5]{4^3} = (\sqrt[5]{4})^3$$

In general, the formula for fractional indices is:

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

More fractional indices

a) $27^{\frac{4}{3}} = (27^{\frac{1}{3}})^4$
$= (\sqrt[3]{27})^4$
$= 3^4$
$= 81$

b) $36^{-\frac{3}{2}} = \frac{1}{36^{\frac{3}{2}}}$
$= \frac{1}{(\sqrt{36})^3}$
$= \frac{1}{6^3}$
$= \frac{1}{216}$

c) $(100x^4y^6)^{\frac{3}{2}} = \left[(100x^4y^6)^{\frac{1}{2}}\right]^3$
$= (\sqrt{100}\,x^2y^3)^3$
$= (10x^2y^3)^3$
$= 1000x^6y^9$

d) $8^{-\frac{2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{3}}$
$= \sqrt[3]{\left(\frac{1}{8}\right)^2}$
$= \sqrt[3]{\frac{1}{64}}$
$= \frac{1}{4}$

1. Use the law for raising indices to indices to rewrite the following in simplest index form:

a $(3^2)^6 =$

b $(4^2)^5 =$

c $(6^7)^8 =$

d $(5^{10})^2 =$

e $(x^3)^4 =$

f $(b^5)^6 =$

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2. Use index laws to rewrite the following in simplest index form:

a $(2x^2)^4 =$

b $(3t^3)^4 =$

c $(xy^3)^5 =$

d $(p^{10}q)^4 =$

e $(a^2b^7)^5 =$

f $(3u^3)^4 =$

g $(4a^2b^7)^4 =$

h $(-2x^4y^5)^4 =$

i $(a^4b^2c^3)^5 =$

j $(3p^3q^4r^7)^4 =$

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3. Use index laws to rewrite the following in simplest index form:

a $\left(\frac{2}{3}\right)^2 =$

b $\left(\frac{x^2}{y^3}\right)^4 =$

c $\left(\frac{2x}{3y}\right)^2 =$

d $\left(\frac{3x^2}{2y^3}\right)^3 =$

e $\left(\frac{3a}{4b}\right)^3 =$

f $\left(\frac{2x^2y}{y}\right)^4 =$

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1. Use the law for raising indices to indices to rewrite the following in simplest index form:

a $(3^2)^6 = 3^{2\times6} = 3^{12}$

b $(4^2)^5 = 4^{2\times5} = 4^{10}$

c $(6^7)^8 = 6^{7\times8} = 6^{56}$

d $(5^{10})^2 = 5^{10\times2} = 5^{20}$

e $(x^3)^4 = x^{3\times4} = x^{12}$

f $(b^5)^6 = b^{5\times6} = b^{30}$

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2. Use index laws to rewrite the following in simplest index form:

a $(2x^2)^4 = 2^4x^{2\times4} = 2^4x^8$

b $(3t^3)^4 = 3^4t^{3\times4} = 3^4t^{12}$

c $(xy^3)^5 = x^5y^{3\times5} = x^5y^{15}$

d $(p^{10}q)^4 = p^{10\times4}q^4 = p^{40}q^4$

e $(a^2b^7)^5 = a^{2\times5}b^{7\times5} = a^{10}b^{35}$

f $(3u^3)^4 = 3^4u^{3\times4} = 3^4u^{12}$

g $(4a^2b^7)^4 = 4^4a^{2\times4}b^{7\times4} = 4^4a^8b^{28}$

h $(-2x^4y^5)^4 = (-2)^4x^{4\times4}y^{5\times4} = (-2)^4x^{16}y^{20}$

i $(a^4b^2c^3)^5 = a^{4\times5}b^{2\times5}c^{3\times5} = a^{20}b^{10}c^{15}$

j $(3p^3q^4r^7)^4 = 3^4p^{3\times4}q^{4\times4}r^{7\times4} = 3^4p^{12}q^{16}r^{28}$

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3. Use index laws to rewrite the following in simplest index form:

a $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$

b $\left(\frac{x^2}{y^3}\right)^4 = \frac{x^{2\times4}}{y^{3\times4}} = \frac{x^8}{y^{12}}$

c $\left(\frac{2x}{3y}\right)^2 = \frac{2^2x^2}{3^2y^2}$

d $\left(\frac{3x^2}{2y^3}\right)^3 = \frac{3^3x^{2\times3}}{2^3y^{3\times3}} = \frac{3^3x^6}{2^3y^9}$

e $\left(\frac{3a}{4b}\right)^3 = \frac{3^3a^3}{4^3b^3}$

f $\left(\frac{2x^2y}{y}\right)^4 = \frac{2^4x^{2\times4}y^4}{y^4} = 2^4x^8\times\frac{y^4}{y^4} = 2^4x^8$

4. Use the zero index law to simplify the following:

a) $4^0 =$

b) $67^0 =$

c) $(362^2)^0 =$

d) $3(4^0) =$

e) $(-5)^0 =$

f) $-5^0 =$

g) $(2\times 10)^0 =$

h) $2\times 10^0 =$

i) $(a^5b^{20}c^7)^0 =$

j) $6p^0 =$

k) $2\times x^0 =$

l) $3\times 4x^0 =$

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5. Simplify the following expressions using the law for negative indices:

a) $10^{-2} =$

b) $5^{-1} =$

c) $(-3)^{-3} =$

d) $-3^{-3} =$

e) $(-4)^{-2} =$

f) $-4^{-2} =$

g) $p^{-7} =$

h) $ab^{-3} =$

i) $6p^{-2} =$

j) $(6p)^{-2} =$

k) $10x^3 \div 10x^7 =$

l) $20p^4q^7 \div 10p^6q^{10} =$

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6. Use fractional indices to write the following in surd form:

a) $7^{\frac{1}{3}} =$

b) $10^{-\frac{1}{4}} =$

c) $n^{\frac{1}{6}} =$

d) $4^{\frac{3}{2}} =$

e) $q^{\frac{6}{7}} =$

f) $x^{-\frac{3}{4}} =$

g) $mn^{\frac{6}{5}} =$

h) $(mn)^{\frac{6}{5}} =$

i) $a^{\frac{2}{5}}b^{\frac{4}{7}} =$

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4. Use the zero index law to simplify the following:

a) $4^0 = 1$

b) $67^0 = 1$

c) $(362^2)^0 = 1$

d) $3(4^0) = 3\times 1$
$= 3$

e) $(-5)^0 = 1$

f) $-5^0 = -1$

g) $(2\times 10)^0 = 20^0$
$= 1$

h) $2\times 10^0 = 2\times 1$
$= 2$

i) $(a^5b^{20}c^7)^0 = 1$

j) $6p^0 = 6\times 1$
$= 6$

k) $2\times x^0 = 2\times 1$
$= 2$

l) $3\times 4x^0 = 3\times 4\times 1$
$= 12$

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5. Simplify the following expressions using the law for negative indices:

a) $10^{-2} = \frac{1}{10^2}$
$= \frac{1}{100}$

b) $5^{-1} = \frac{1}{5^1}$
$= \frac{1}{5}$

c) $(-3)^{-3} = \frac{1}{(-3)^3}$
$= \frac{1}{(-3)\times(-3)\times(-3)}$
$= -\frac{1}{27}$

d) $-3^{-3} = \frac{1}{-3^3}$
$= \frac{1}{-(3\times 3\times 3)}$
$= -\frac{1}{27}$

e) $(-4)^{-2} = \frac{1}{(-4)^2}$
$= \frac{1}{16}$

f) $-4^{-2} = \frac{-1}{4^2}$
$= \frac{-1}{4\times 4}$
$= -\frac{1}{16}$

g) $p^{-7} = \frac{1}{p^7}$

h) $ab^{-3} = \frac{a}{b^3}$

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5. Simplify the following expressions using the law for negative indices:

a) $6p^{-2} = \frac{6}{p^2}$

b) $(6p)^{-2} = \frac{1}{(6p)^2} = \frac{1}{36p^2}$

c) $10x^3 \div 10x^7 = \frac{10x^3}{10x^7} = x^{3-7} = x^{-4} = \frac{1}{x^4}$

d) $20p^4q^7 \div 10p^6q^{10} = \frac{20p^4q^7}{10p^6q^{10}} = 2p^{4-6}q^{7-10} = 2p^{-2}q^{-3} = \frac{2}{p^2q^3}$

6. Use fractional indices to write the following in surd form:

a) $7^{\frac{1}{3}} = (7)^{\frac{1}{3}} = \sqrt[3]{7}$

b) $10^{-\frac{1}{4}} = \frac{1}{10^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{10}}$

c) $n^{\frac{1}{6}} = \sqrt[6]{n}$

d) $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = \sqrt[2]{4^3}$

e) $q^{\frac{6}{7}} = (q^6)^{\frac{1}{7}} = \sqrt[7]{q^6}$

f) $x^{-\frac{3}{4}} = \frac{1}{x^{\frac{3}{4}}}$

g) $mn^{\frac{6}{5}} = m(n^6)^{\frac{1}{5}} = m\sqrt[5]{n^6}$

h) $(mn)^{\frac{6}{5}} = \sqrt[5]{(mn)^6} = \sqrt[5]{m^6n^6}$

i) $a^{\frac{2}{5}}b^{\frac{4}{7}} = (a^2)^{\frac{1}{5}}(b^4)^{\frac{1}{7}} = \sqrt[5]{a^2}\times\sqrt[7]{b^4} = \sqrt[5]{a^2}\,\sqrt[7]{b^4}$

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7. Write the following in index notation:

a) $\sqrt{8} = 8^{\frac{1}{2}}$

b) $\sqrt{x} = x^{\frac{1}{2}}$

c) $\sqrt[3]{10} = 10^{\frac{1}{3}}$

d) $\frac{1}{\sqrt{21}} = \frac{1}{21^{\frac{1}{2}}} = 21^{-\frac{1}{2}}$

e) $\frac{1}{\sqrt[4]{10}} = \frac{1}{10^{\frac{1}{4}}} = 10^{-\frac{1}{4}}$

f) $\frac{1}{\sqrt[5]{x}} = \frac{1}{x^{\frac{1}{5}}} = x^{-\frac{1}{5}}$

g) $\sqrt[y]{x} = x^{\frac{1}{y}}$

7. Write the following in index notation:

a) $\sqrt{8} =$

b) $\sqrt{x} =$

c) $\sqrt[3]{10} =$

d) $\frac{1}{\sqrt{21}} =$

e) $\frac{1}{\sqrt[4]{10}} =$

f) $\frac{1}{\sqrt[5]{x}} =$

g) $\sqrt[y]{x} =$

8. Write the following in simplest positive index notation:

a) $(8^{\frac{1}{3}})^2 =$

b) $\left(\frac{1}{4}\right)^{-2} =$

c) $w^4 \times \sqrt{w} =$

d) $(27^{\frac{2}{3}}) \times (27^2)^{\frac{1}{6}} =$

e) $p^{\frac{1}{2}} \div p^{\frac{3}{2}} =$

f) $\sqrt[4]{x^7} \times \sqrt[4]{x^3} =$

g) $\sqrt[5]{y^3} \div \sqrt[5]{y^{13}} =$

h) $(\sqrt[7]{pq^3})^0 =$

i) $x \times \left(\frac{1}{2}\right)^0 \times \sqrt[3]{10} \times x =$

j) $(216^2)^{\frac{1}{6}} \div 4^{\frac{1}{2}} =$

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5. Simplify the following expressions using the law for negative indices:

a) $6p^{-2} = \frac{6}{p^2}$

b) $(6p)^{-2} = \frac{1}{(6p)^2}$
$= \frac{1}{36p^2}$

c) $10x^3 \div 10x^7 = \frac{10x^3}{10x^7}$
$= x^{3-7}$
$= x^{-4}$
$= \frac{1}{x^4}$

d) $20p^4q^7 \div 10p^6q^{10} = \frac{20p^4q^7}{10p^6q^{10}}$
$= 2p^{4-6}q^{7-10}$
$= 2p^{-2}q^{-3}$
$= \frac{2}{p^2q^3}$

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6. Use fractional indices to write the following in surd form:

a) $7^{\frac{1}{3}} = (7)^{\frac{1}{3}}$
$= \sqrt[3]{7}$

b) $10^{-\frac{1}{4}} = \frac{1}{10^{\frac{1}{4}}}$
$= \frac{1}{\sqrt[4]{10}}$

c) $n^{\frac{1}{6}} = \sqrt[6]{n}$

d) $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}}$
$= \sqrt[2]{4^3}$

e) $q^{\frac{6}{7}} = (q^6)^{\frac{1}{7}}$
$= \sqrt[7]{q^6}$

f) $x^{-\frac{3}{4}} = \frac{1}{x^{\frac{3}{4}}}$

g) $mn^{\frac{6}{5}} = m(n^6)^{\frac{1}{5}}$
$= m\sqrt[5]{n^6}$

h) $(mn)^{\frac{6}{5}} = \sqrt[5]{(mn)^6}$
$= \sqrt[5]{m^6n^6}$

i) $a^{\frac{2}{5}}b^{\frac{4}{7}} = (a^2)^{\frac{1}{5}}(b^4)^{\frac{1}{7}}$
$= \sqrt[5]{a^2} \times \sqrt[7]{b^4}$
$= \sqrt[5]{a^2}\,\sqrt[7]{b^4}$

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7. Write the following in index notation:

a) $\sqrt{8} = 8^{\frac{1}{2}}$

b) $\sqrt{x} = x^{\frac{1}{2}}$

c) $\sqrt[3]{10} = 10^{\frac{1}{3}}$

d) $\frac{1}{\sqrt{21}} = \frac{1}{21^{\frac{1}{2}}}$
$= 21^{-\frac{1}{2}}$

e) $\frac{1}{\sqrt[4]{10}} = \frac{1}{10^{\frac{1}{4}}}$
$= 10^{-\frac{1}{4}}$

f) $\frac{1}{\sqrt[5]{x}} = \frac{1}{x^{\frac{1}{5}}}$
$= x^{-\frac{1}{5}}$

g) $\sqrt[y]{x} = x^{\frac{1}{y}}$

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8. Write the following in simplest positive index notation:

a) $(8^{\frac{1}{3}})^2 = \sqrt[3]{8^2}$
$= \sqrt[3]{8^2}$
$= \sqrt[3]{64}$
$= 4$

b) $\left(\frac{1}{4}\right)^{-2} = \frac{1}{\left(\frac{1}{4}\right)^2}$
$= \frac{1}{\left(\frac{1}{16}\right)}$
$= 16$

c) $w^4 \times \sqrt{w} = w^4 \times w^{\frac{1}{2}}$
$= w^{(4+\frac{1}{2})}$
$= w^{\frac{9}{2}}$

d) $(27^{\frac{2}{3}}) \times (27^2)^{\frac{1}{6}} = 27^{\frac{2}{3}} \times 27^{2\times\frac{1}{6}}$
$= 27^{\frac{2}{3}} \times 27^{\frac{2}{6}}$
$= 27^{\frac{2}{3}} \times 27^{\frac{1}{3}}$
$= 27^{(\frac{2}{3}+\frac{1}{3})}$
$= 27^{\frac{3}{3}}$
$= 27^1$
$= 27$

e) $p^{\frac{1}{2}} \div p^{\frac{3}{2}} = p^{\frac{1}{2}-\frac{3}{2}}$
$= p^{-1}$
$= \frac{1}{p}$

f) $\sqrt[4]{x^7} \times \sqrt[4]{x^3} = x^{\frac{7}{4}} \times x^{\frac{3}{4}}$
$= x^{(\frac{7}{4}+\frac{3}{4})}$
$= x^{\frac{10}{4}}$

g) $\sqrt[5]{y^3} \div \sqrt[5]{y^{13}} = y^{\frac{3}{5}} \div y^{\frac{13}{5}}$
$= y^{\frac{3}{5}-\frac{13}{5}}$
$= y^{-\frac{10}{5}}$
$= y^{-2}$
$= \frac{1}{y^2}$

h) $(\sqrt[7]{pq^3})^0 = 1$

i) $x \times \left(\frac{1}{2}\right)^0 \times \sqrt[3]{10} \times x = x \times 1 \times 10^{\frac{1}{3}} \times x$
$= 10^{\frac{1}{3}} \times x \times x$
$= 10^{\frac{1}{3}}x^2$

j) $(216^2)^{\frac{1}{6}} \div 4^{\frac{1}{2}} = 216^{2\times\frac{1}{6}} \div 4^{\frac{1}{2}}$
$= 216^{\frac{2}{6}} \div \sqrt{4}$
$= 216^{\frac{1}{3}} \div 2$
$= \sqrt[3]{216} \div 2$
$= 6 \div 2$
$= 3$