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REVIEW TOPICS (SP3)

MATHEMATICAL INDUCTION

CEM – Yr 12 – Mathematical Induction – Review Booklet – SP3

1. Prove by mathematical induction that
$$1(2)^2 + 2(3)^2 + \ldots\ldots + n(n+1)^2 = \frac{n}{12}(n+1)(n+2)(3n+5)$$
for positive integers $n \geq 1$.

2. The n th term of a series is given by $T_n = \frac{1}{(2n-1)(2n+1)}$

(i) Find T_5 and T_{k+1}

(ii) Assuming that the sum S_k of the first k terms of this series is given by
$S_k = \frac{k}{2k+1}$, prove that $S_{k+1} = \frac{k+1}{2k+3}$

(iii) Using the results of parts (i) and (ii) and the Principle of Mathematical Induction, show that the sum of the first
n terms of the series is $\frac{n}{2n+1}$

3. Prove by Mathematical Induction that $5^n \geq 1+4n$

4. Using mathematical induction, show that $a^n - 1$ is divisible by $a - 1$, for n a positive integer.

5. Use the Principle of Mathematical Induction to prove that

$$5^n > 3^n + 2^n \text{ for integers } n > 1$$

6. Show by Mathematical Induction that

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+----+\frac{1}{4n^2-1}=\frac{n}{2n+1} \qquad \text{for all } n \geq 1$$

Answers

1. (d) Step 1 $n=1$

$1\times 2^2 = \frac{1}{12}(1+1)(1+2)(3+5)$

$= \frac{48}{12}$

$= 4 \quad \therefore$ True for $n=1$

Step 2 $n=k$ assume True.

Let

$1(2)^2 + 2(3)^2 + \dots + k(k+1)^2 = \frac{k}{12}(k+1)(k+2)(3k+5)$

& so when $n=k+1$

$S_{k+1} = 1(2)^2 + 2(3)^2 + \dots + k(k+1)^2 + (k+1)([k+1]+1)^2$

$S_{k+1} = \frac{k}{12}(k+1)(k+2)(3k+5) + (k+1)([k+1]+1)^2$

$= \frac{k}{12}(k+1)(k+2)(3k+5) + (k+1)(k+2)(k+2)$ ✓

$= \frac{(k+1)}{12}(k+2)\left[k(3k+5) + 12(k+2)\right]$

$= \frac{(k+1)}{12}(k+2)\left[3k^2 + 5k + 12k + 24\right]$

$= \frac{(k+1)(k+2)}{12}\left[3k^2 + 17k + 24\right]$ ✓

$= \frac{k+1}{12}([k+1]+1)\left[(k+3)(3k+8)\right]$

$= \frac{k+1}{12}([k+1]+1)([k+1]+2)(3[k+1]+5)$ ✓

LHS = RHS QED

Hence if it is true for $n=1$, + $n=2$ and so on, then it is true for all n, n is positive number

2. (Q7) a) (i) $T_5 = \frac{1}{(2(5)-1)(10+1)}$ ②

$= \frac{1}{99}$ ✓

(iii) $T_{k+1} = \frac{1}{(2k+1)(2k+3)}$

(ii) ③

$S_{k+1} = S_k + T_{k+1}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$ ✓

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3}$ ✓

(iii) ③ Let $n=1$

$T_1 = \frac{1}{(2-1)(2+1)} = \frac{1}{3}$ $\qquad S_1 = \frac{1}{2(1)+1} = \frac{1}{3}$

$\therefore$ true for $n=1$ ✓

assume true for $n=k$

ie $S_k = \frac{k}{2k+1}$

Prove true for $n=k+1$

ie $S_k + T_{k+1} = S_{k+1}$

LHS $= \frac{k+1}{2k+3}$ from (ii) ✓

RHS $= \frac{k}{2k+1}$ = LHS

∴ if true for $n = k$
then true for $n = k+1$
But it is true for $n = 1$
∴ it is true for $n = 1+1 = 2$
& since true for 2 it is
true for $n = 3$ & so on
∴ $S_n = \frac{n}{2n+1}$ is proven true
(03) ✓ by M.I. ✓

3. b)
(4)
Prove true for $n = 1$

$5^n \geq 1 + 4n$

$5^1 = 5 \qquad 1 + 4(1) = 5$ ✓

∴ $5^1 \geq 1 + 4(1)$
ie true for $n = 1$

Assume true for $n = k$
ie $5^k \geq 1 + 4k$

Prove true for $n = k+1$ ✓
ie $5^{k+1} \geq 1 + 4(k+1)$
$5^{k+1} \geq 5 + 4k$

using assumption
$5^k \geq 1 + 4k$
×5
$5 \times 5^k \geq 5(1+4k)$
∴ $5^{k+1} \geq 5 + 20k$ ✓
but $5 + 20k \geq 1 + 4k$
ie $16k \geq -4$
∴ $5^{k+1} \geq 1 + 4k$

∴ if true for .... ✓
See (iii) above

4. a) $a^n - 1$ ÷ by $a-1$, $n \geq 0$

Test $n=1$

$a^n - 1 = a-1$ which is divisible by $a-1$

∴ result true for $n=1$

Assume result true for $n=k$

$a^k - 1 = (a-1)M$ where M is a constant

Hence show result true for $n=k+1$

$a^{k+1} - 1 = (a-1)P$

$a^{k+1} - 1 = a^{k+1} - a + a - 1$

$= a(a^k - 1) + 1(a-1)$

$= a(a-1)M + 1(a-1)$

$= (a-1)(aM+1)$

$= (a-1)P$ where $P = aM+1$

Hence if result true for $n=k$ then it is true for $n=k+1$

Since result true for $n=1$, then true for $n=2$, $n=3$ and so on for all +ve integers of n.

Finish

5. Let S_n be the statement $5^n > 2^n + 3^n$ for integers $n > 1$

S_2: $5^2 > 2^2 + 3^2$ is true

Assume S_k i.e. assume that $5^k > 2^k + 3^k$ for integers k

Then $5^{k+1} = 5 \times 5^k > 5 \times (2^k + 3^k)$ since S_k is true

i.e. $5^{k+1} > 5 \times 2^k + 5 \times 3^k$

so $5^{k+1} > 2 \times 2^k + 3 \times 2^k + 3 \times 3^k + 2 \times 3^k$

$5^{k+1} > [2^{k+1} + 3^{k+1}] + 3 \times 2^k + 2 \times 3^k$

i.e. $5^{k+1} > 2^{k+1} + 3^{k+1}$

Therefore, if S_k is true, then S_{k+1} is true.

But S_2 is true so S_3 is true and so on for all integer values of $n > 1$

6. Step 1

Let $n = 1$.

LHS $\frac{1}{4-1} = \frac{1}{3}$

RHS $\frac{n}{2n+1} = \frac{1}{3} = \text{LHS}$

$\therefore$ true for $n = 1$

Step 2

Assume true for $n = k$.

$\frac{1}{3} + \frac{1}{15} + \dots \frac{1}{4k^2-1} = \frac{k}{2k+1}$

R.T.P. also true for $n = k+1$

$\frac{1}{3} + \frac{1}{15} + \dots \frac{1}{4(k+1)^2-1} = \frac{k+1}{2k+3}$

LHS $\frac{1}{3} + \frac{1}{15} + \dots \frac{1}{4k^2-1} + \frac{1}{4(k+1)^2-1} = \frac{k}{2k+1} + \frac{1}{4(k+1)^2-1}$

(from assumption)

$= \frac{k}{2k+1} + \frac{1}{4k^2+8k+3}$ ✓

$= \frac{k}{2k+1} + \frac{1}{(2k+3)(2k+1)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)}$ ✓

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3}$ ✓ $= \text{RHS}$

Step 3

If true for $n = k$ and $n = k+1$ and also true for $n = 1$, then it is true for $n = 1+1 = 2$, $n = 2+1 = 3$ and so on.

$\therefore$ by the Principle of Mathematical Induction, it is true for all real $n \geq 1$. ✓