CANDIDATE NUMBER

Sydney Grammar School

2017 Assessment Examination

FORM VI

MATHEMATICS EXTENSION 1

Monday 22nd May 2017

General Instructions

- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 125 boys

Examiner
REJ

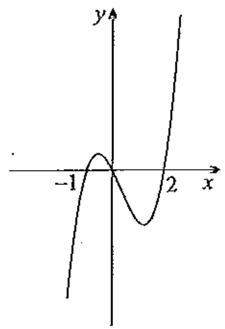
SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

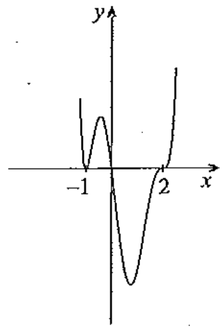
QUESTION ONE

Which diagram best represents the graph of $y = x(x-2)^3(x+1)^2$?

(A)



(B)



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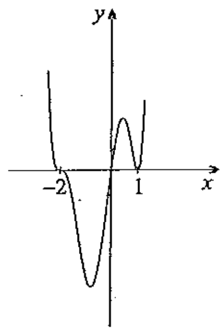
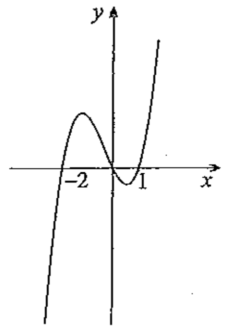


(D)



QUESTION TWO

A particle is moving in simple harmonic motion. The displacement function is given by $x = 2\sin 3t$. What is the amplitude A and the period of motion T?

(A) $A = 2$ and $T = \frac{2\pi}{3}$

(B) $A = 2$ and $T = \frac{\pi}{3}$

(C) $A = 3$ and $T = \pi$

(D) $A = 3$ and $T = \frac{\pi}{2}$

QUESTION THREE

Three stones are projected simultaneously off the edge of a tall vertical cliff. Stone A is projected horizontally at $10\,\text{ms}^{-1}$; stone B is projected horizontally at $20\,\text{ms}^{-1}$; and stone C is dropped from rest. The only force acting on the stones is acceleration due to gravity.

Which statement is correct?

(A) Stone A reaches the ground first.

(B) Stone B reaches the ground first.

(C) Stone C reaches the ground first.

(D) All three stones reach the ground at the same time.

QUESTION FOUR

A polynomial equation has roots α, β and γ where

$$\alpha+\beta+\gamma = 2,\ \alpha\beta+\alpha\gamma+\beta\gamma = 5 \text{ and } \alpha\beta\gamma = -2.$$

Which of the following polynomial equations could it be?

(A) $x^3 + 2x^2 + 5x - 2 = 0$

(B) $x^3 - 2x^2 + 5x + 2 = 0$

(C) $x^3 - 2x^2 - 5x + 2 = 0$

(D) $x^3 - 2x^2 + 5x - 2 = 0$

Examination continues overleaf ...

QUESTION FIVE

Suppose $\tan 2\theta = 1$, where $0 \le \theta \le 2\pi$. In which quadrant(s) could θ lie?

(A) the first quadrant

(B) the first or second quadrant

(C) the first or third quadrant

(D) the first, second, third or fourth quadrant

QUESTION SIX

Which of the following is the derivative of $\tan^{-1} 3x$?

(A) $\frac{3}{1-9x^2}$

(B) $\frac{3}{\sqrt{1-9x^2}}$

(C) $\frac{3}{1+9x^2}$

(D) $\frac{3}{9+x^2}$

QUESTION SEVEN

Consider the function $f(x) = \frac{2x}{x+1}$ and its inverse function $f^{-1}(x)$.

What is the value of $f^{-1}(4)$?

(A) 2

(B) $\frac{8}{5}$

(C) $\frac{5}{8}$

(D) -2

QUESTION EIGHT

When the polynomial $P(x)$ is divided by $x^2 - 4$, the remainder is $2x - 1$.
What is the remainder when $P(x)$ is divided by $x - 2$?

(A) -5

(B) -1

(C) 3

(D) 5

Examination continues next page ...

QUESTION NINE

What is the value of k such that $\displaystyle\int_0^k \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{3}$?

(A) $\sqrt{2}$

(B) 1

(C) $\sqrt{3}$

(D) $\dfrac{\sqrt{3}}{2}$

QUESTION TEN

Suppose

$$P(x) = x^4 + 5x^3 + 3x^2 - 6x - 10,\ Q(x) = x^3 + 6x^2 + 8x - 2$$

and

$$P(x) - Q(x) = (x+4)(x-2)(x+1)^2.$$

What are the geometric implications of this?

(A) $P(x)$ and $Q(x)$ are tangent at $x=-4$, $x=2$ and $x=-1$.

(B) $P(x)$ and $Q(x)$ cross at $x=-4$, $x=2$ and $x=-1$.

(C) $P(x)$ and $Q(x)$ cross at $x=-4$, $x=2$ and are tangent at $x=-1$.

(D) $P(x)$ and $Q(x)$ are tangent at $x=-4$, $x=2$ and cross at $x=-1$.

——— End of Section I ———

Examination continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Write down the exact value of the following.

(i) $\tan^{-1}\sqrt{3}$ $\boxed{1}$

(ii) $\cos^{-1}\sin\left(\dfrac{4\pi}{3}\right)$ $\boxed{1}$

(b) Find the remainder when $P(x) = x^3 - 3x^2 + 3x - 5$ is divided by $x-1$. $\boxed{2}$

(c) Find the equation of the tangent to $y = \cos^{-1}(2x)$ at the point $\left(0, \dfrac{\pi}{2}\right)$. $\boxed{3}$

(d) A particle's displacement function is given by $x = 4\sin(2t) + 1$, where displacement x is in metres and time t is in seconds.

(i) Show that the particle moves in simple harmonic motion by showing that $\ddot{x} = -n^2(x - x_0)$. $\boxed{1}$

(ii) Write down the centre of motion of the particle. $\boxed{1}$

(iii) Between what two values of x does the particle oscillate? $\boxed{1}$

(iv) Find the particle's speed when $x = 5$. $\boxed{1}$

(e) Consider the polynomial equation $x^3 + 2x^2 - 3x + 5 = 0$. If the roots of the equation are α, β and γ, find the value of the following.

(i) $\alpha + \beta + \gamma$ $\boxed{1}$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ $\boxed{1}$

(iii) $\alpha\beta\gamma$ $\boxed{1}$

(iv) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ $\boxed{1}$

Examination continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Divide $P(x) = 2x^3 + 3x^2 + 12x + 16$ by $D(x) = x + 3$ and express your answer in the form $P(x) = D(x)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder. **2**

(b) (i) Prove that $(\cos x + \sin x)^2 = 1 + \sin 2x$. **1**

(ii) Hence find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \sin 2x}\, dx$. **2**

(c) The polynomial $P(x) = x^3 - 6x^2 + kx + 14$ has a zero at $x = 1$.

(i) Show that the value of k is -9. **1**

(ii) Fully factorise $P(x)$. **2**

(iii) Sketch $y = x^3 - 6x^2 - 9x + 14$. **1**

(iv) Hence solve $x^3 - 6x^2 - 9x + 14 > 0$. **1**

(d) A ball is projected horizontally from the top of a vertical cliff 120 m above the ground with initial velocity $30\,\text{ms}^{-1}$. Let acceleration due to gravity be $10\,\text{ms}^{-2}$ downward and the base of the cliff be the origin. You may ignore air resistance.

(i) Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, derive equations for the horizontal and vertical components of displacement. **2**

(ii) How far from the base of the cliff does the ball land? Leave your answer in exact form. **2**

(iii) What is the exact speed of the ball when it hits the ground? **1**

Examination continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) A particle is moving horizontally in simple harmonic motion about the origin O. At time t seconds it has displacement x metres from O given by $x = \cos 2t - \sin 2t$.

(i) Express x in the form $R\cos(2t + \alpha)$ for $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Find the amplitude and period of the motion. **1**

(iii) Find the maximum velocity of the particle. **2**

(b) A particle moves with velocity $v = (1 - x)^2\,\text{ms}^{-1}$, as a function of displacement x. Initially the particle is at the origin.

(i) Find an expression for acceleration in terms of x. **1**

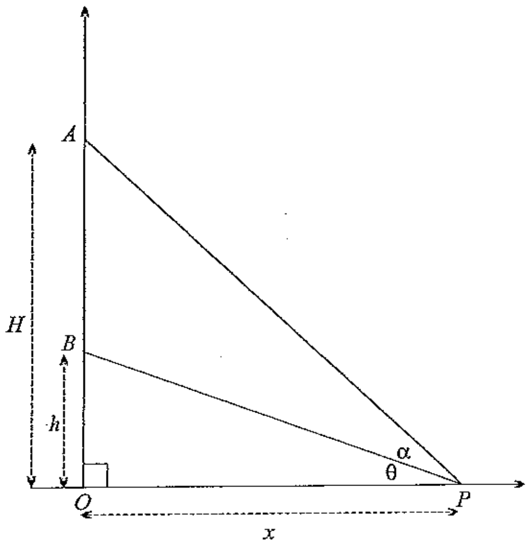
(ii) Show that $t = \dfrac{1}{1-x} - 1$. **2**

(iii) Find the time taken for the particle to slow down to a speed of 1% of its initial speed. **2**

Examination continues next page ...

QUESTION THIRTEEN (Continued)

(c) The diagram below shows the point P on the horizontal axis, a variable distance x from the origin O. The points A and B are fixed points on the vertical axis, with distances H and h respectively from the origin O.



Let $\angle BPO = \theta$ and $\angle APB = \alpha$.

(i) Show that $\alpha = \tan^{-1}\left(\dfrac{H}{x}\right) - \tan^{-1}\left(\dfrac{h}{x}\right)$. $\boxed{1}$

(ii) Show that α is a maximum when $x^2 = Hh$. You do not need to justify that it is a maximum. $\boxed{2}$

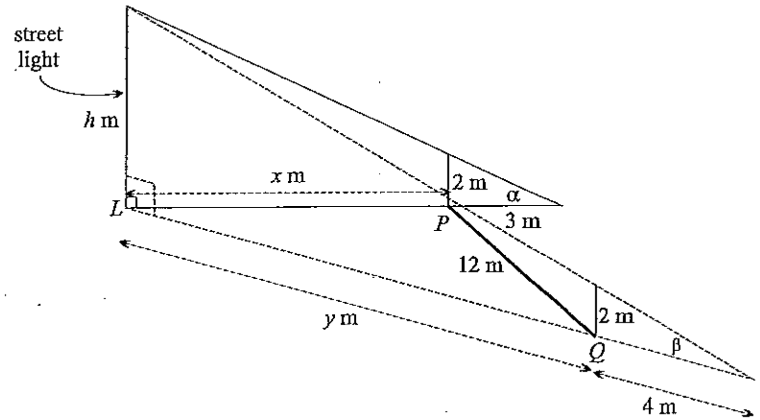
(iii) Using the expansion of $\tan(A - B)$, show that the maximum value of α occurs for $\tan\alpha = \dfrac{\sqrt{Hh}(H-h)}{2Hh}$. $\boxed{2}$

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. Marks

(a) (i) Find the derivative of the function $x\cos^{-1}x - \sqrt{1-x^2}$. $\boxed{2}$

(ii) Hence evaluate $\displaystyle\int_0^1 \cos^{-1}x\,dx$. $\boxed{2}$

(b) A two metre tall man casts a shadow by a street light h metres high at L. At P, x metres due East of L, his shadow is 3 metres long. He walks 12 metres on a bearing of 150°T to a point Q. At Q, y metres from L, his shadow is 4 metres long. The angles α and β are shown on the diagram.



(i) Write an equation for x in terms of h. $\boxed{1}$

(ii) Write an equation for y in terms of h. $\boxed{1}$

(iii) Find the height of the street light correct to one decimal place. $\boxed{3}$

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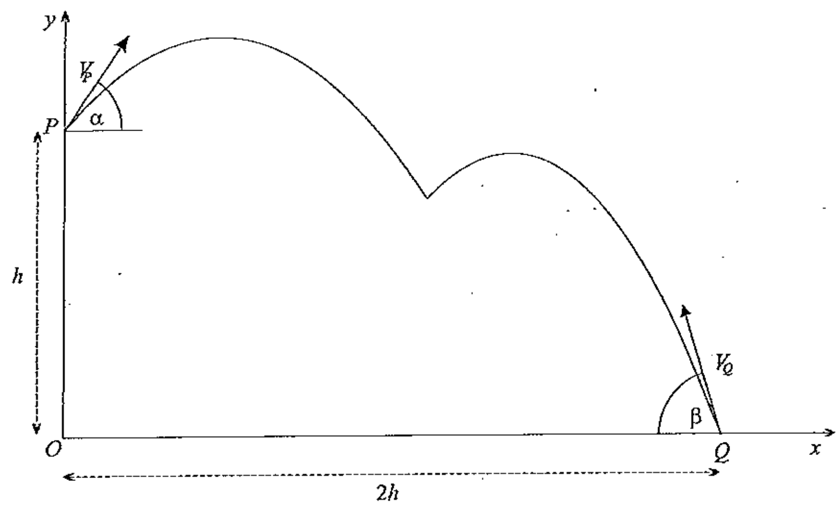
QUESTION FOURTEEN (Continued)

(c) Two points O and Q are $2h$ metres apart on horizontal ground. The point P is h metres directly above O. A particle is projected from P towards Q with speed $V_P\,\mathrm{ms}^{-1}$, inclined at an angle α above the horizontal. At the same time another particle is projected from Q towards P with speed $V_Q\,\mathrm{ms}^{-1}$, inclined at an angle β above the horizontal. The two particles collide T seconds after projection.



(i) For the particle launched from P, the acceleration equations are $\ddot{x}_P = 0$ and $\ddot{y}_P = -g$. Show that at time t seconds, the respective horizontal and vertical distances from O, x_P and y_P are: **[2]**

$$x_P = V_P t\cos\alpha$$
$$y_P = -\frac{1}{2}gt^2 + V_P t\sin\alpha + h.$$

(ii) Write down the respective horizontal and vertical distances from O for the particle launched from Q. **[2]**

(iii) Show that $\dfrac{V_P}{V_Q} = \dfrac{2\sin\beta - \cos\beta}{2\sin\alpha + \cos\alpha}$. **[2]**

End of Section II

END OF EXAMINATION

Extension 1 Solutions

Question 1

$y = x(x-2)^3(x+1)^2$

double zero at $x=-1$
triple zero at $x=2$

(B)

Question 2

$x = 2\sin 3t$
amp $= 2$ $\quad T = \frac{2\pi}{3}$

(A)

Question 3

The only force acting is in the vertical direction so all stones will land at the same time

(D)

Question 4

$\alpha+\beta+\gamma = \frac{-b}{a}$ $\quad \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$ $\quad \alpha\beta\gamma = \frac{-d}{a}$

$b=-2 \quad c=5 \quad d=2$
$a=1$

$x^3 - 2x^2 + 5x + 2 = 0$ (B)

Question 5

$\tan 2\theta = 1$ $\qquad 0 \le \theta \le 2\pi$
$\qquad 0 \le 2\theta \le 4\pi$

$2\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
Q1 Q2 Q3 Q4

(D)

Question 6

$\frac{d}{dx}\tan^{-1}3x$

$= \frac{3}{1+9x^2}$

(C)

Question 7

$f(x)$: $\quad y = \frac{2x}{x+1}$

$f^{-1}(x)$ $\quad x = \frac{2y}{y+1}$

$xy + x = 2y$
$x = 2y - xy$

$x = y(2-x)$

$y = \frac{x}{2-x}$

$f^{-1}(x) = \frac{x}{2-x}$

$f^{-1}(4) = \frac{4}{2-4}$
$= -2$

(D)

Question 8

$\frac{P(x)}{x-2} = \frac{(x-2)(x+2)Q(x)}{x-2} + \frac{2x-1}{x-2}$

$$\begin{array}{r} 2 \\ x-2\overline{)\,2x-1} \\ \underline{2x-4} \\ 3 \end{array}$$

$\therefore$ Remainder is 3

alternatively $R(x) = 2x-1$
$R(2) = 2\times 2 - 1$
$= 3$

(C)

Question 9

$\int_0^k \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{3}$

$\int_0^k \frac{1}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^k$

$= \sin^{-1}\left(\frac{k}{2}\right) - \sin^{-1}(0)$

$\sin^{-1}\left(\frac{k}{2}\right) = \frac{\pi}{3}$

$\frac{k}{2} = \frac{\sqrt{3}}{2}$

$k = \sqrt{3}$ (C)

Question 10

Double zero at $x=-1$ tells us the curves are tangent
single zero at $x=-4$ & $x=2$
indicates the curves cross

(C)

Question 11

a) i) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ ✓

ii) $\cos^{-1}\sin\left(\frac{4\pi}{3}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{5\pi}{6}$ ✓

b) By remainder theorem
Remainder $= P(1)$

$P(1) = 1^3 - 3\times1^2 + 3\times1 - 5$ ✓
$= -4$

The remainder is -4 ✓

c) $y = \cos^{-1}(2x)$

$\frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$ ✓

at $x=0$ $m_T = \frac{-2}{\sqrt{1-0}}$

$= -2$ ✓

$y - \frac{\pi}{2} = -2(x-0)$ ✓

$y = -2x + \frac{\pi}{2}$

d) i) $x = 4\sin 2t + 1$
$\dot{x} = 8\cos 2t$
$\ddot{x} = -16\sin 2t$
$\ddot{x} = -4(4\sin 2t + 1 - 1)$

$\ddot{x} = -4(x-1)$ ✓ show
which is in the form $-n^2(x - x_0)$
So the particle is in SHM.

ii) The centre of motion is $x = 1$ m. ✓

iii) The amplitude is 4 so the particle oscillates between $x=-3$ and $x=5$ ✓

iv) $x=5$ is an extremity
$\therefore$ the speed is zero or $0\,ms^{-1}$ ✓

e) $x^3 + 2x^3 - 3x + 5 = 0$
a b c d

i) $\alpha + \beta + \gamma = \frac{-b}{a}$
$= -2$ ✓

ii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$
$= -3$ ✓

iii) $\alpha\beta\gamma = \frac{-d}{a}$
$= -5$ ✓

iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= \frac{-3}{-5}$

$= \frac{3}{5}$ ✓

/15

Question 12

$$
\begin{array}{r}
2x^2 - 3x + 21 \\
x+3 \,\overline{)\, 2x^3 + 3x^2 + 12x + 16} \\
\underline{2x^3 + 6x^2 \quad\quad\quad\quad} \\
-3x^2 + 12x + 16 \\
\underline{-3x^2 - 9x \quad\quad} \\
21x + 16 \\
\underline{21x + 63} \\
-47
\end{array}
$$

✓ long division (solid attempt)

✓ result

$P(x) = (x+3)(2x^2 - 3x + 21) \quad -47$

b) i) $LHS = (\cos x + \sin x)^2$

$= \cos^2 x + 2\sin x \cos x + \sin^2 x$

$= 1 + 2\sin x \cos x$

$= 1 + \sin 2x$ ✓ show (expansion)

$= RHS$ as required

so $(\cos x + \sin x)^2 = 1 + \sin 2x$

ii) $\int_0^{\pi/4} \sqrt{1 + \sin 2x}\, dx$

$= \int_0^{\pi/4} \sqrt{(\cos x + \sin x)^2}\, dx$

from part i)

$= \int_0^{\pi/4} \cos x + \sin x \, dx$

$= \left[\sin x - \cos x\right]_0^{\pi/4}$ ✓

$= \left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right] - \left[\sin 0 - \cos 0\right]$

$= \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - \left[0 - 1\right]$

$= 1$ ✓

c) i) $P(x) = x^3 - 6x^2 + kx + 14$

$P(1) = 0$

$0 = 1^3 - 6 \times 1^2 + k + 14$

$0 = k + 9$

$k = -9$ as required ✓

ii) $P(2) = -20$

$P(-2) = 0$

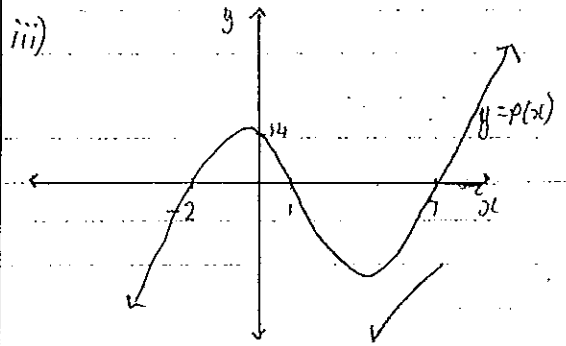
$P(7) = 0$ ✓ or division

$P(-7) =$

The factors are $(x+2)$ and $(x-7)$

$P(x) = (x-1)(x+2)(x-7)$ ✓

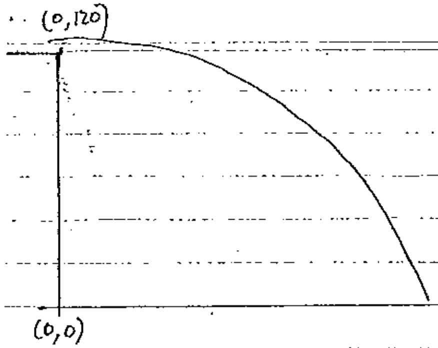
iii)



✓

iv) $\{x : -2 < x < 1 \text{ or } x > 7\}$ ✓

from my graph.



$t = 0 \quad \dot{x} = 30 \quad \dot{y} = 0$

$\quad\quad x = 0 \quad y = 120$

i) $\ddot{x} = 0 \qquad \ddot{y} = -10$

$\dot{x} = \int 0\, dt \qquad \dot{y} = \int -10\, dt$

$= C_1 \qquad = -10t + C_3$

at $t = 0$ $\dot{x} = 30 \qquad t = 0$ $\dot{y} = 0$ $C_3 = 0$

$\dot{x} = 30 \qquad \dot{y} = -10t$

$x = \int 30\, dt \qquad y = \int -10t\, dt$

$= 30t + C_2 \qquad = -5t^2 + C_4$

$t = 0$ $x = 0$ $C_2 = 0 \qquad t = 0$ $y = 120$ $C_4 = 120$

$\underline{x = 30t}$ ✓ $\qquad \underline{y = -5t^2 + 120}$ ✓

must be derived, penalise for omitting constants

ii) we want $y = 0$

$-5t^2 + 120 = 0$

$5t^2 = 120$

$t^2 = 24$

$t = \pm 2\sqrt{6}$ s $(t > 0)$

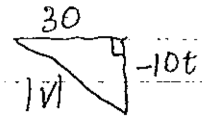
when $t = 2\sqrt{6}$ ✓ $x = 30 \times 2\sqrt{6}$

$= 60\sqrt{6}$ m ✓

accept $t = \sqrt{24}$.

The ball lands $60\sqrt{6}$ m from

iii)



at $t = 2\sqrt{6}$

$\dot{x} = 30 \qquad \dot{y} = -20\sqrt{6}$

$V^2 = 30^2 + (20\sqrt{6})^2$

$V^2 = 3300$

$|V| = 10\sqrt{33}$ ms^{-1}

The speed when the ball lands is $10\sqrt{33}$ ms^{-1} ✓

(accept $\sqrt{3300}$)

Question 13

a) i) $x = R\cos(2t+\alpha)$

$x = R\cos 2t \cos\alpha - R\sin 2t \sin\alpha$

So equating coefficients.

$R\cos\alpha = 1 \qquad R\sin\alpha = 1$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1+1$

$R^2 = 2$

$R = \pm\sqrt{2} \quad R>0$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1}$

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

✓ ✓

$x = \sqrt{2}\cos(2t+\frac{\pi}{4})$

ii) amplitude is $\sqrt{2}$ metres.

Period is $\frac{2\pi}{2} = \pi$ seconds ✓

iii) maximum velocity

$\dot{x} = -2\sqrt{2}\sin(2t+\frac{\pi}{4})$ ✓ differentiate

maximum velocity is $2\sqrt{2}\ ms^{-1}$ ✓

b) i) $v = (1-x)^2$

$v^2 = (1-x)^4$

$\ddot{x} = \frac{d}{dx}\frac{1}{2}v^2$

$= \frac{d}{dx}\frac{(1-x)^4}{2}$

$= \frac{4(1-x)^3}{2} \times -1$

$\ddot{x} = -2(1-x)^3$ ✓

Alternate working

$\ddot{x} = v \cdot \frac{dv}{dx}$

$= (1-x)^2 \times 2(1-x) \times (-1)$

$= -2(1-x)^3$

ii) $v = (1-x)^2$

$\frac{dx}{dt} = (1-x)^2$

$\frac{dt}{dx} = (1-x)^{-2}$

$t = \int (1-x)^{-2}\,dx$ ✓ express as $\frac{dt}{dx}$ & integral

$= \frac{(1-x)^{-1}}{-1\times -1} + C$

$t = \frac{1}{1-x} + C$

$t=0 \quad x=0$

$0 = 1 + C$ ✓ evaluate constant

$C = -1$

$t = \frac{1}{1-x} - 1$ as required

iii) at $t=0$ $x=0$ so initially $v = (1-0)^2$

$= 1$

We want $v = 1\%$ of 1

$= 0{\cdot}01\ ms^{-1}$

$0{\cdot}01 = (1-x)^2$

$1-x = \pm 0{\cdot}1$

$x = 0{\cdot}9$ or $1{\cdot}1$

When $x = 0{\cdot}9$ ✓

$t = \frac{1}{1-0{\cdot}9} - 1$

$= 9$

or $t = -11 \quad t>0$

It takes 9 seconds to reach 1% of initial velocity ✓

(alternate working accepted)

c) In ΔBOP:

$\tan\theta = \frac{h}{x}$

$\theta = \tan^{-1}\frac{h}{x}$

In ΔAOP

$\tan(\alpha+\theta) = \frac{H}{x}$

$\alpha+\theta = \tan^{-1}\left(\frac{H}{x}\right)$

$\alpha = \tan^{-1}\left(\frac{H}{x}\right) - \theta$

✓ show $\alpha = \tan^{-1}\left(\frac{H}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right)$

as required.

ii) To maximise α, solve $\frac{d\alpha}{dx} = 0$

$\alpha = \tan^{-1}\left(\frac{H}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right)$

$\frac{d\alpha}{dx} = \frac{\frac{-H}{x^2}}{1+\frac{H^2}{x^2}} - \frac{\frac{-h}{x^2}}{1+\frac{h^2}{x^2}}$

(multiply through by x^2)

$= \frac{-H}{x^2+H^2} + \frac{h}{x^2+h^2}$

Solve $= 0$

$\frac{h}{x^2+h^2} - \frac{H}{x^2+H^2} = 0$

✓ differentiate & equate to zero

$hx^2 + hH^2 - Hx^2 - Hh^2 = 0$

$x^2(h-H) = Hh^2 - hH^2$

$x^2 = \frac{Hh(h-H)}{(h-H)}$

simplify expression

$x^2 = Hh$ as required

iii) $\alpha = \tan^{-1}\left(\frac{H}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right)$

$\tan\alpha = \frac{\frac{H}{x} - \frac{h}{x}}{1 + \frac{H}{x}\times\frac{h}{x}}$ expand double angle

$\tan\alpha = \frac{Hx - hx}{x^2 + Hh}$

$\tan\alpha = \frac{x(H-h)}{Hh + Hh}$ $[x^2 = Hh$ part ii)$]$

$\tan\alpha = \frac{\sqrt{Hh}(H-h)}{2Hh}$ $[x = \sqrt{Hh}$ part ii)$]$

sub + simplify

as required

Question 14

a) $\frac{d}{dx}\, x\cos^{-1}x - \sqrt{1-x^2}$

$u = x \quad v = \cos^{-1}x$

$u' = 1 \quad v' = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\, x\cos^{-1}x - \sqrt{1-x^2}$ product rule chain rule

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$

$= \cos^{-1}x$

ii) $\int_0^1 \cos^{-1}x$

$= \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$

$= \left[1\cos^{-1}1 - \sqrt{1-1^2}\right] - \left[0 - \sqrt{1-0}\right]$

$= [0-0] - [0-1]$

$= 1$

b) i) $\tan\alpha = \frac{h}{x+3}$

$\tan\alpha = \frac{2}{3}$

equating

$\frac{2}{3} = \frac{h}{x+3}$

$2x + 6 = 3h$

$2x = 3h - 6$

$x = \frac{3h-6}{2}$

ii) $\tan\beta = \frac{h}{y+4}$

$\tan\beta = \frac{2}{4} = \frac{1}{2}$

equating

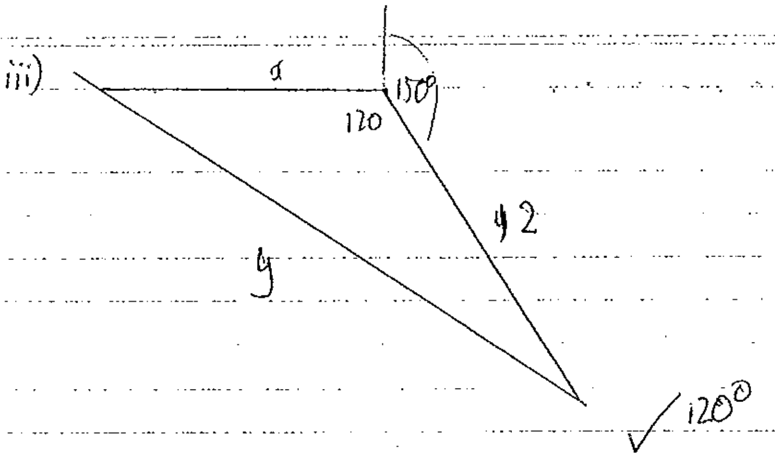
$\frac{h}{y+4} = \frac{1}{2}$

$2h = y + 4$

$y = 2h - 4$

Accept similar triangle working

iii)



$$y^2 = x^2 + 12^2 - 2x \times 12 \cos 120$$ ✓ cosine rule.

$$(2h-4)^2 = \left(\frac{3h-6}{2}\right)^2 + 144 - 24\left(\frac{3h-6}{2}\right)\left(-\frac{1}{2}\right)$$

$$(2h-4)^2 = \frac{(3h-6)^2}{4} + 144 + 6(3h-6)$$

$$4(2h-4)^2 = (3h-6)^2 + 576 + 24(3h-6)$$

$$4[4h^2-16h+16] = 9h^2 - 36h + 36 + 576 + 72h - 144$$

$$16h^2 - 64h + 64 = 9h^2 + 36h + 468$$

$$7h^2 - 100h - 404 = 0$$

$$h = \frac{100 \pm \sqrt{100^2 - 4 \times 7 \times -404}}{2 \times 7} \quad h > 0$$

$$h \simeq 17 \cdot 5704$$
$$h \simeq 17 \cdot 6m \text{ (1dp)}$$ ✓

At P:

c) initially $x_1 = 0$ $y_1 = 2h$
$\dot{x}_1 = V_p \cos\alpha$ $\dot{y}_1 = V_p \sin\alpha$

$$\ddot{x} = 0$$
$$\dot{x} = \int 0\, dt$$
$$= C_1$$
$t=0$ $\dot{x} = V_p \cos\alpha$
$$\dot{x} = V_p \cos\alpha$$
$$x = \int V_p \cos\alpha$$
$$= V_p t \cos\alpha + C_2$$
$t=0$ $x=0$
$$\boxed{x = V_p t \cos\alpha}$$ ✓

$$\ddot{y} = -g$$
$$\dot{y} = \int -g\, dt$$
$$\dot{y} = -gt + C_3$$
$t=0$ $\dot{y} = V_p \sin\alpha$
$$\dot{y} = -gt + V_p \sin\alpha$$
$$y = \int -gt + V_p \sin\alpha \, dt$$
$$y = \frac{-gt^2}{2} + V_p t \sin\alpha + C_4$$
$t=0$ $y=h$
$$\boxed{y = \frac{-gt^2}{2} + V_p t \sin\alpha + h}$$ ✓

Show

ii) $x_Q = -V_Q t\cos\beta + 2h$ ✓ $y_Q = -\frac{gt^2}{2} + V_Q t\sin\beta$ ✓

iii) at $t = T$ $x_Q = x_P$ $y_Q = y_P$

$-V_Q T\cos\beta + 2h = V_P T\cos\alpha$

$2h = V_P T\cos\alpha + V_Q T\cos\beta$

$-\frac{gT^2}{2} + V_Q T\sin\beta = -\frac{gT^2}{2} + V_P T\sin\alpha + h$

$h = V_Q T\sin\beta - V_P T\sin\alpha$

$2h = 2V_Q T\sin\beta - 2V_P T\sin\alpha$

✓ progress toward eliminating h.

equating

$V_P T\cos\alpha + V_Q T\cos\beta = 2V_Q T\sin\beta - 2V_P T\sin\alpha$

$V_P\cos\alpha + 2V_P\sin\alpha = 2V_Q\sin\beta - V_Q\cos\beta$

$V_P(\cos\alpha + 2\sin\alpha) = V_Q(2\sin\beta - \cos\beta)$

$$\frac{V_P}{V_Q} = \frac{2\sin\beta - \cos\beta}{2\sin\alpha + \cos\alpha}$$ ✓ sub in T & simplify.

as required.

alternate working accepted