CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2017 Assessment Examination

FORM VI

MATHEMATICS EXTENSION 2

Thursday 18th May 2017

General Instructions
- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 70 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 60 Marks
- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist
- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 73 boys

Examiner
LRP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The ellipse $9x^2 + 16y^2 = 144$ has eccentricity $\dfrac{\sqrt{7}}{4}$. What are the coordinates of its foci? [1]

(A) $S(0, \sqrt{7})$ and $S'(0, -\sqrt{7})$

(B) $S(\sqrt{7}, 0)$ and $S'(-\sqrt{7}, 0)$

(C) $S(4\sqrt{7}, 0)$ and $S'(-4\sqrt{7}, 0)$

(D) $S(0, 4\sqrt{7})$ and $S'(0, -4\sqrt{7})$

QUESTION TWO

What is the remainder when $P(z) = 2z^3 - 3z^2 + 4z - 2$ is divided by $(z + i)$? [1]

(A) $1 - 2i$

(B) $1 - 6i$

(C) $1 + 2i$

(D) $1 + 6i$

QUESTION THREE

Every point on a certain conic is twice as far from the line $x = 4$ as from the point $(1, 0)$. What is a possible equation of the conic? [1]

(A) $\dfrac{x^2}{3} - \dfrac{y^2}{4} = 1$

(B) $\dfrac{x^2}{4} - \dfrac{y^2}{3} = 1$

(C) $\dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$

(D) $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

Examination continues next page ...

QUESTION FOUR

Two of the zeroes of the polynomial $P(x) = x^4 - 4x^3 + 9x^2 - 16x + 20$ are $a + ib$ and $2ib$, where a and b are real and $b \neq 0$. What is the value of a? $\boxed{1}$

(A) 2

(B) -2

(C) 4

(D) -4

QUESTION FIVE

Which of the following is equivalent to $\int_a^b x^3 e^{2x^4}\, dx$, where a and b are real constants? $\boxed{1}$

(A) $\int_{a^4}^{b^4} e^{2u}\, du$

(B) $\frac{1}{8}\int_a^b e^u\, du$

(C) $\frac{1}{4}\int_{a^4}^{b^4} e^{2u}\, du$

(D) $\frac{1}{8}\int_{8a^3}^{8b^3} e^u\, du$

QUESTION SIX

Let x metres be the displacement of a particle of mass 1000 kilograms from the origin on a straight path. The particle experiences a constant propelling force of 10 000 newtons and a resistive force of magnitude $100v^2$ newtons, where v is the velocity of the particle at time t seconds. What is the equation of motion of the particle? $\boxed{1}$

(A) $\ddot{x} = 10\,000 - 100v^2$

(B) $\ddot{x} = 10 - 0{\cdot}1v^2$

(C) $\ddot{x} = 10\,000 - 0{\cdot}1v^2$

(D) $\ddot{x} = 10 - 100v^2$

Examination continues overleaf ...

QUESTION SEVEN

Let $x = \sin\theta - \cos\theta$ and $y = \frac{1}{2}\sin 2\theta$. What is the correct expression for $\frac{dy}{dx}$? $\boxed{1}$

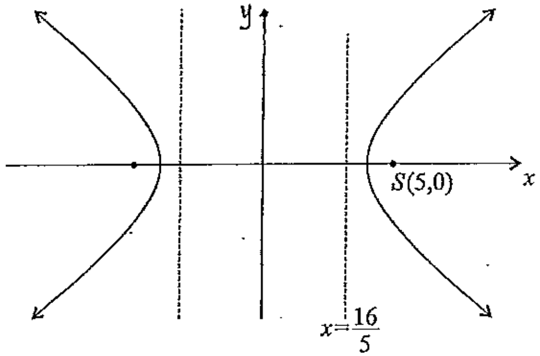
(A) $\cos\theta - \sin\theta$

(B) $\sec\theta + \operatorname{cosec}\theta$

(C) $\sec\theta - \operatorname{cosec}\theta$

(D) $\cos\theta + \sin\theta$

QUESTION EIGHT $\boxed{1}$



A hyperbola centred at the origin has a focus at $S(5,0)$ and a directrix $x = \frac{16}{5}$. What is the eccentricity of the hyperbola?

(A) $\frac{4}{3}$

(B) $\frac{25}{16}$

(C) $\frac{5}{4}$

(D) 4

QUESTION NINE

Which of the following is equivalent to $\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} x \sin x \, dx$? [1]

(A) 0

(B) $2\int_0^{\pi} x \sin x \, dx$

(C) $2\int_0^{\frac{\pi}{2}} x \sin x \, dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x \sin x \, dx$

(D) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x \sin x \, dx$

QUESTION TEN

Consider the relation $a^2x^2 + (1-a^2)y^2 = b^2$, where a and b are non-zero real numbers. [1]

Which of the following CANNOT be represented by the relation?

(A) a circle

(B) a parabola

(C) a hyperbola

(D) a pair of straight lines

End of Section I

SECTION II - Written Response

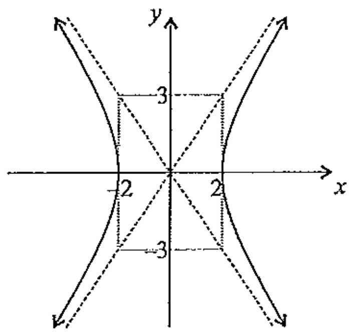
Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. Marks

(a)



The diagram shows a hyperbola with asymptotes $y = \frac{3x}{2}$ and $y = -\frac{3x}{2}$.

(i) Write an equation for the hyperbola. [1]

(ii) Find the eccentricity of the hyperbola. [1]

(iii) Write down the coordinates of both foci of the hyperbola. [1]

(iv) Write down the equations of both directrices of the hyperbola. [1]

(b) Consider the polynomial $P(x) = 3x^3 - 10x^2 + 7x + 10$.

(i) Given that one zero of $P(x)$ is $2 - i$, find the other two zeroes. [2]

(ii) Hence express $P(x)$ as the product of a linear factor and a quadratic factor, both with real coefficients. [2]

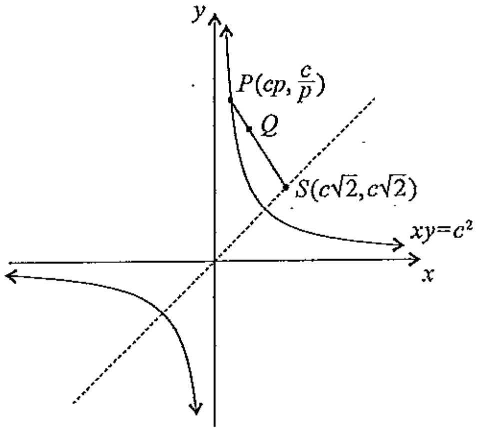
(c) The polynomial equation $2x^3 - 9x^2 + 12x - 4 = 0$ has a double root at $x = \alpha$.

(i) Find the value of α. [2]

(ii) Find the remaining root. [1]

QUESTION ELEVEN (Continued)

(d)



The point $P\left(cp, \dfrac{c}{p}\right)$, where $p > 0$, lies on the rectangular hyperbola $xy = c^2$ with focus $S\left(c\sqrt{2}, c\sqrt{2}\right)$. The point Q divides the interval PS in the ratio $1 : 2$.

(i) Show that the coordinates of Q are $\left(\dfrac{2cp + c\sqrt{2}}{3}, \dfrac{2c + cp\sqrt{2}}{3p}\right)$. **2**

(ii) Find the Cartesian equation of the locus of Q as P varies. **2**

QUESTION TWELVE (15 marks) Use a separate writing booklet. Marks

(a) When a polynomial $P(x)$ is divided by $(x-2)$ and $(x-3)$ the respective remainders are 4 and 3. Determine what the remainder must be when $P(x)$ is divided by $(x-2)(x-3)$. **3**

(b) Barbara decides to go bungee jumping. This involves being tied to a bridge by an elastic cable of unstretched length d metres and falling vertically from rest from this point. After Barbara free falls d metres, she will be slowed down by the cable, which exerts a resistive force proportional to the distance greater than d that she has fallen.

If we take the origin at bridge level, x to be the distance fallen in metres and g to be the acceleration due to gravity in ms^{-2}, then Barbara's motion during her initial descent will be defined by:

$$\ddot{x} = \begin{cases} g & \text{when } x \leq d \\ g - k(x-d) & \text{when } x > d \end{cases}$$

Let Barbara's speed be $v\,\text{ms}^{-1}$.

(i) Find an expression for v^2 at the instant when Barbara first passes $x = d$. **2**

(ii) Hence show that $v^2 = 2gx - k(x-d)^2$ for $x > d$. **2**

(c) A ball is thrown vertically upwards with an initial velocity of $7\sqrt{6}\,\text{ms}^{-1}$. It is subject to gravity and air resistance. The acceleration of the ball is given by $\ddot{x} = -\left(9{\cdot}8 + 0{\cdot}1v^2\right)$, where x metres is its vertical displacement from the point of launch and $v\,\text{ms}^{-1}$ is its velocity at time t seconds.

(i) Find an expression for time t as a function of velocity v. **3**

(ii) Hence find the time taken for the ball to reach its maximum height. Give your answer correct to three significant figures. **1**

(iii) Find an expression for vertical displacement x in terms of velocity v. **3**

(iv) Hence find the maximum height reached. Give your answer in exact form. **1**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The rise and fall in sea level due to tides can be modelled with simple harmonic motion. On a certain day, a channel is 8 metres deep at 7 am when it is low tide, and 14 metres deep at 2 pm when it is high tide.

(i) Sketch a graph showing the depth of the water d at time t. Write an equation that models the depth of water d as a function of time t. Take the origin of time to correspond to the low tide at 7 am. **3**

(ii) A ship must sail down the channel at some time between 7 am and 9 pm. If the ship requires a water depth of at least 12 metres, between what times of day can the ship pass safely through? Give your answer correct to the nearest minute. **3**

(b) The roots of $2x^3 - 9x^2 + 8x - 2 = 0$ are α, β and γ.

(i) Find the value of $\alpha\beta\gamma$. **1**

(ii) Hence find a simplified cubic polynomial equation with integer coefficients that has roots $\dfrac{\alpha\beta}{\gamma}$, $\dfrac{\alpha\gamma}{\beta}$, and $\dfrac{\beta\gamma}{\alpha}$. **3**

(c) The equation $x^3 - 3ax + b = 0$, with real constants $a > 0$ and $b \neq 0$, has three distinct real roots.

(i) Find the stationary points of $y = x^3 - 3ax + b$ in terms of a and b and determine their nature. **3**

(ii) Hence show that $b^2 < 4a^3$, explaining your reasoning carefully. **2**

Examination continues overleaf ...

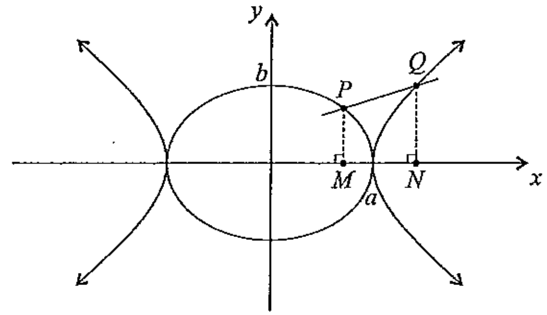
QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $I_n = \displaystyle\int_0^1 \frac{1}{(x^2+1)^n}\,dx$ for any integer $n \geq 1$.

(i) Show that $I_{n+1} = \dfrac{1}{2n}\left[2^{-n} + (2n-1)\,I_n\right]$. **4**

(ii) Hence evaluate I_3. **2**

(b)



Distinct points $P(a\cos\theta, b\sin\theta)$ and $Q(a\sec\theta, b\tan\theta)$ lie on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ respectively, as shown, where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. The points M and N are the feet of the perpendiculars from P and Q respectively to the x-axis.

(i) The line PQ meets the x-axis at K. Show that $\dfrac{KM}{KN} = \cos\theta$. **1**

(ii) Hence find the coordinates of K. **2**

(iii) Show that the tangent to the ellipse at P has equation $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ and deduce that it passes through N. **3**

(iv) The tangent to the hyperbola at Q has equation $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ and passes through M. Do NOT prove this. Let T be the point of intersection of PN and QM.

(α) Show that T always lies on the same vertical line and state its equation. **1**

(β) Where on this line can T lie? Justify your answer. **1**

(γ) Suppose that θ is now in the second or third quadrant. Explain where T may lie. **1**

End of Section II

END OF EXAMINATION

EXTENSION 2 - SOLUTIONS

May Assessment 2017

Q1. $\frac{9x^2}{144} + \frac{16y^2}{144} = \frac{144}{144}$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

$a = 4$, $e = \frac{\sqrt{7}}{4}$ $\therefore ae = \sqrt{7}$

$\therefore$ Foci: $(\pm\sqrt{7}, 0)$ B

Q2. $P(-i) = 2(-i)^3 - 3(-i)^2 + 4(-i) - 2$

$= 2i + 3 - 4i - 2$

$= 1 - 2i$ A

Q3. $e = \frac{1}{2}$ $\therefore$ ellipse

Focus: $(1, 0)$ & vertical directrix

D

Q4. Roots must be $a+ib$, $a-ib$, $2ib$, $-2ib$

Sum of roots: $2a = \frac{-(-4)}{1}$

$\therefore a = 2$

A

Q5. $\int_a^b x^3 e^{2x^4}\,dx$ Let $u = x^4$

$du = 4x^3\,dx$

$= \frac{1}{4}\int_a^b 4x^3 e^{2x^4}\,dx$

x	a	b
u	a^4	b^4

$= \frac{1}{4}\int_{a^4}^{b^4} e^{2u}\,du$

C

Q6. $100v^2$ N ⟵ • ⟶ $10\,000$ N

$1000\ddot{x} = 10000 - 100v^2$

$\ddot{x} = 10 - 0.1v^2$ B

Q7. $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= \frac{\cos 2\theta}{\cos\theta + \sin\theta}$

$= \frac{\cos^2\theta - \sin^2\theta}{\cos\theta + \sin\theta}$

$= \frac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{\cos\theta + \sin\theta}$

$= \cos\theta - \sin\theta$ A

Q8. $ae = 5$ — ①

$\frac{a}{e} = \frac{16}{5}$ — ②

① × ②: $a^2 = 16$

$a = 4$

$\therefore e = \frac{5}{4}$ C

Q9. $f(-x) = -x \times \sin(-x)$
$= -x \times -\sin x$
$= x \sin x$
$= f(x)$ $\therefore$ even

C

Q10. $a^2x^2 + (1-a^2)y^2 = b^2$

circle ✓ $a^2 = 1 - a^2$
$\therefore a^2 = \frac{1}{2}$

hyperbola ✓ $1 - a^2 < 0$
$a^2 > 1$

straight lines ✓ $a^2 = 0 \rightarrow y = \pm b$ (but a is non-zero anyway...)
$a^2 = 1 \rightarrow x = \pm b$

$\therefore$ parabola

B

QUESTION 11:

a) i) $a = 2$
$b = 3$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$ ✓

ii) $b^2 = a^2(e^2 - 1)$
$9 = 4(e^2 - 1)$

$e^2 = \frac{13}{4}$

$\therefore e = \frac{\sqrt{13}}{2}$ ✓

iii) $ae = 2 \times \frac{\sqrt{13}}{2}$
$= \sqrt{13}$

$\therefore$ foci: $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$ ✓

iv) $\frac{a}{e} = \frac{2}{\frac{\sqrt{13}}{2}}$
$= \frac{4}{\sqrt{13}}$

$\therefore$ directrices: $x = \frac{4}{\sqrt{13}}$ or $x = -\frac{4}{\sqrt{13}}$ ✓

[OR $x = \pm\frac{4\sqrt{13}}{13}$]

b) $P(x) = 3x^3 - 10x^2 + 7x + 10$

i) $\overline{2-i} = 2+i$ must also be a zero ✓

Let α be the third zero.

From sum of zeroes:

$$\alpha + 2 + i + 2 - i = \frac{10}{3}$$

$$\therefore \alpha = -\frac{2}{3}$$ ✓

OR/ From product of zeroes:

$$\alpha(2+i)(2-i) = -\frac{10}{3}$$

$$5\alpha = -\frac{10}{3}$$

$$\therefore \alpha = -\frac{2}{3}$$

ii) $P(x) = 3\left(x + \frac{2}{3}\right)\left(x - (2-i)\right)\left(x - (2+i)\right)$ ✓

$= (3x+2)(x^2 - 4x + 5)$ ✓

c) i) Let $P(x) = 2x^3 - 9x^2 + 12x - 4$

$P'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x-1)(x-2)$

$\therefore P'(x) = 0$ when $x = 1$ or $x = 2$ ✓

$P(1) = 1 \quad \therefore x = 1$ is not the double root

$P(2) = 0 \quad \therefore x = 2$ is the double root

$\therefore \alpha = 2$ ✓

ii) Let β be the other root.

From sum of roots:

$$2 + 2 + \beta = \frac{9}{2}$$

$\therefore \beta = \frac{1}{2}$ $\quad \therefore$ the remaining root is $x = \frac{1}{2}$ ✓

OR/ From product of roots:

$$2 \times 2 \times \beta = \frac{4}{2}$$

$$\therefore \beta = \frac{1}{2}$$

d) i) $m = 1, \ n = 2$

$P\left(cp, \frac{c}{p}\right) \rightarrow (x_1, y_1)$

$S(c\sqrt{2}, c\sqrt{2}) \rightarrow (x_2, y_2)$

$$x_Q = \frac{mx_2 + nx_1}{m+n}$$

$$= \frac{1 \times c\sqrt{2} + 2 \times cp}{1+2}$$ ✓

$$= \frac{c\sqrt{2} + 2cp}{3} = \frac{2cp + c\sqrt{2}}{3}$$

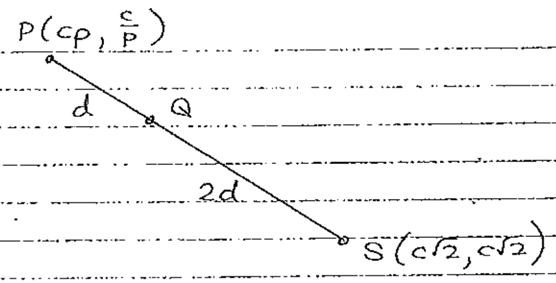
$$y_Q = \frac{my_2 + ny_1}{m+n}$$

$$= \frac{1 \times c\sqrt{2} + 2 \times \frac{c}{p}}{1+2} \times \frac{p}{p}$$

$$= \frac{cp\sqrt{2} + 2c}{3p} = \frac{2c + cp\sqrt{2}}{3p}$$ ✓

$$\therefore Q\left(\frac{2cp + c\sqrt{2}}{3}, \frac{2c + cp\sqrt{2}}{3p}\right)$$

OR



$$x_Q = x_P + \frac{x_S - x_P}{3}$$

$$= cp + \frac{c\sqrt{2} - cp}{3}$$

$$= \frac{2cp + c\sqrt{2}}{3}$$

$$y_Q = y_P - \frac{y_P - y_S}{3}$$

$$= \frac{c}{p} - \frac{\frac{c}{p} - c\sqrt{2}}{3}$$

$$= \frac{3c - (c - cp\sqrt{2})}{3p}$$

$$= \frac{2c + cp\sqrt{2}}{3p}$$

ii) $x = \dfrac{2cp + c\sqrt{2}}{3}$ (from (i))

$$p = \frac{3x - c\sqrt{2}}{2c} \quad \text{--- (1)}$$

$$y = \frac{2c + cp\sqrt{2}}{3p}$$

$$3yp - cp\sqrt{2} = 2c$$

$$p = \frac{2c}{3y - c\sqrt{2}} \quad \text{--- (2)}$$

(1) = (2):

$$\frac{3x - c\sqrt{2}}{2c} = \frac{2c}{3y - c\sqrt{2}}$$

$$(3x - c\sqrt{2})(3y - c\sqrt{2}) = 4c^2$$

QUESTION 12:

a) $P(x) = Q(x) \times D(x) + R(x)$

$= Q(x)(x-2)(x-3) + ax + b$

$P(2) = 4$:

$4 = 2a + b$ ①

$P(3) = 3$:

$3 = 3a + b$ ②

② − ①: $a = -1$

Sub into ①: $4 = -2 + b$

$b = 6$

∴ the remainder is $-x + 6$ (or $6 - x$)

b) i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g$

$\frac{1}{2}v^2 = gx + C$

when $x = 0$, $v = 0$:

$0 = 0 + C \rightarrow C = 0$

$\therefore \frac{1}{2}v^2 = gx$

$v^2 = 2gx$ (must show calc. of constant)

when $x = d$:

$v^2 = 2gd$.

ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - k(x-d)$ [notice that this equation is also valid when $x = d$]

$\frac{1}{2}v^2 = gx - \frac{k(x-d)^2}{2} + C_1$

$v^2 = 2gx - k(x-d)^2 + C_2$

when $x = d$, $v^2 = 2gd$:

$2gd = 2gd - k(d-d)^2 + C_2$

$\therefore C_2 = 0$

$\therefore v^2 = 2gx - k(x-d)^2$ as required.

c) $t = 0$

$v = 7\sqrt{6}$ m/s

$\ddot{x} = -(9.8 + 0.1v^2)$

↑+

i) $\frac{dv}{dt} = -(9.8 + 0.1v^2)$

$\frac{dt}{dv} = -\frac{1}{9.8 + 0.1v^2}$

$t = -10\int \frac{1}{98 + v^2}\,dv$

$= -\frac{10}{\sqrt{98}}\tan^{-1}\frac{v}{\sqrt{98}} + C$

when $t = 0$, $v = 7\sqrt{6}$:

$0 = -\frac{10}{\sqrt{98}}\tan^{-1}\sqrt{3} + C$

$\therefore C = \frac{10}{\sqrt{98}} \times \frac{\pi}{3}$

$= \frac{10\pi}{21\sqrt{2}}$

$\therefore t = -\frac{10}{7\sqrt{2}}\tan^{-1}\frac{v}{7\sqrt{2}} + \frac{10\pi}{21\sqrt{2}}$

$\left(\text{or } t = -\frac{5\sqrt{2}}{7}\tan^{-1}\frac{v}{7\sqrt{2}} + \frac{5\pi\sqrt{2}}{21}\right)$

ii) $t = ?$ when $v = 0$:

$$t = 0 + \frac{10\pi}{21\sqrt{2}}$$

$= 1.06$ seconds (to 3 sig. fig.)

iii) $v\frac{dv}{dx} = -(9.8 + 0.1v^2)$

$$\frac{dx}{dv} = -\frac{v}{9.8 + 0.1v^2}$$

$$x = -\frac{10}{2}\int \frac{2v}{98 + v^2}\,dv$$

$$= -5\ln(98 + v^2) + C$$

when $x = 0$, $v = 7\sqrt{6}$:

$$0 = -5\ln(98 + 294) + C$$

$\therefore C = 5\ln 392$

$$x = -5\ln(98 + v^2) + 5\ln 392$$

$$= 5\ln\frac{392}{98 + v^2}$$

iv) $x = ?$ when $v = 0$:

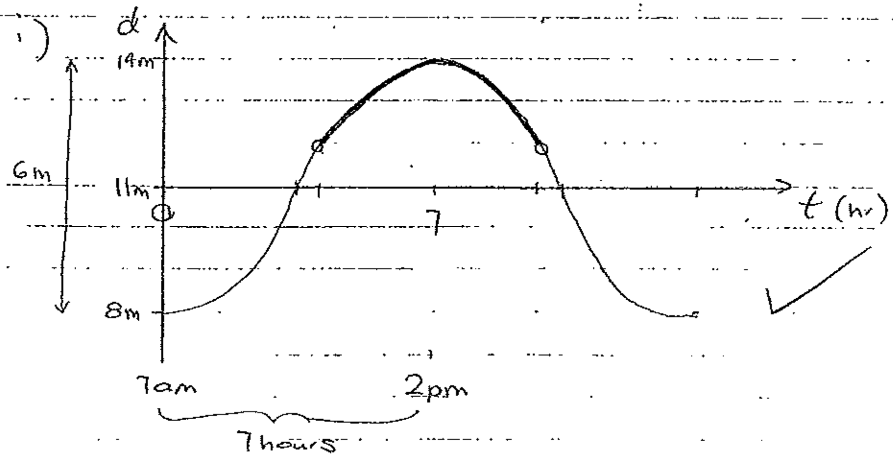
$$x = 5\ln\frac{392}{98}$$

$$= 5\ln 4$$

$= 10\ln 2$ metres

QUESTION 13:

a) i)



$a = 3$

$T = 2 \times 7$
$= 14$

$$T = \frac{2\pi}{n}$$

$$14 = \frac{2\pi}{n}$$

$$\therefore n = \frac{\pi}{7}$$

$$\therefore d = -3\cos\frac{\pi}{7}t + 11$$

ii) $12 = -3\cos\frac{\pi}{7}t + 11$

$$\cos\frac{\pi}{7}t = -\frac{1}{3}$$

$\frac{\pi}{7}t = \pi - \cos^{-1}(\frac{1}{3})$ or $\pi + \cos^{-1}(\frac{1}{3})$

$\therefore t = \frac{7}{\pi}(\pi - \cos^{-1}(\frac{1}{3}))$ or $\frac{7}{\pi}(\pi + \cos^{-1}(\frac{1}{3}))$

$= 4.2572\ldots$ $\quad = 9.7427\ldots$

$\doteqdot 4$ hr 15 min $\quad \doteqdot 9$ hr 45 min

$\therefore$ It can pass through between 11:15am + 4:45pm

b) i) $\alpha\beta\gamma = -\frac{(-2)}{2}$

$= 1$ ✓

ii) $\frac{\alpha\beta}{\gamma} = \frac{\alpha\beta\gamma}{\gamma^2}$ $\quad \frac{\alpha\gamma}{\beta} = \frac{\alpha\beta\gamma}{\beta^2}$ $\quad \frac{\beta\gamma}{\alpha} = \frac{\alpha\beta\gamma}{\alpha^2}$

$= \frac{1}{\gamma^2}$ $\quad = \frac{1}{\beta^2}$ $\quad = \frac{1}{\alpha^2}$

$\therefore$ the required roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ ✓

$\therefore$ replace x with $\frac{1}{\sqrt{x}}$:

$$2\left(\frac{1}{\sqrt{x}}\right)^3 - 9\left(\frac{1}{\sqrt{x}}\right)^2 + 8\left(\frac{1}{\sqrt{x}}\right) - 2 = 0$$ ✓

$$2 - 9\sqrt{x} + 8x - 2x\sqrt{x} = 0$$

$$(2+8x)^2 = \left[\sqrt{x}(2x+9)\right]^2$$

$$4 + 32x + 64x^2 = 4x^3 + 36x^2 + 81x$$

$$4x^3 - 28x^2 + 49x - 4 = 0$$ ✓

c) i) $y = x^3 - 3ax + b$

$$\frac{dy}{dx} = 3x^2 - 3a$$

$$\frac{d^2y}{dx^2} = 6x$$

St. points:

$\frac{dy}{dx} = 0$ when $3x^2 = 3a$

$x = \pm\sqrt{a}$ ✓

When $x = \sqrt{a}$, $y = a\sqrt{a} - 3a\sqrt{a} + b$

$= -2a\sqrt{a} + b$

$\frac{d^2y}{dx^2} = 6\sqrt{a}$

> 0 $\therefore$ minimum turning point at $(\sqrt{a}, -2a\sqrt{a} + b)$ ✓

When $x = -\sqrt{a}$ $y = -a\sqrt{a} + 3a\sqrt{a} + b$

$= 2a\sqrt{a} + b$

$\frac{d^2y}{dx^2} = -6\sqrt{a}$

< 0 $\therefore$ maximum turning point at $(-\sqrt{a}, 2a\sqrt{a} + b)$ ✓

ii) 3 distinct real roots

$\therefore$ turning points must be on opposite sides of the x-axis. ✓

$\therefore y_{max} \times y_{min} < 0$

$(2a\sqrt{a} + b)(-2a\sqrt{a} + b) < 0$ ✓

$b^2 - 4a^3 < 0$

$b^2 < 4a^3$

QUESTION 14:

a) $I_n = \int_0^1 \frac{1}{(x^2+1)^n}\,dx$

i) $I_n = \int_0^1 \frac{d}{dx}(\underset{u}{x}) \times (\underset{v}{x^2+1})^{-n}\,dx$ ✓

$= \left[x(x^2+1)^{-n}\right]_0^1 - \int_0^1 x \times -2nx(x^2+1)^{-n-1}\,dx$

$= 2^{-n} + 2n\int_0^1 \frac{x^2}{(x^2+1)^{n+1}}\,dx$ ✓

$= 2^{-n} + 2n\int_0^1 \frac{x^2+1-1}{(x^2+1)^{n+1}}\,dx$

$= 2^{-n} + 2n\int_0^1 \left(\frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}\right)dx$ ✓

$= 2^{-n} + 2n\left[I_n - I_{n+1}\right]$

$2nI_{n+1} = 2^{-n} + 2nI_n - I_n$ ✓

$\therefore I_{n+1} = \frac{1}{2n}\left[2^{-n} + (2n-1)I_n\right]$

ii) $I_1 = \int_0^1 \frac{1}{x^2+1}\,dx$

$= \left[\tan^{-1}x\right]_0^1$

$= \frac{\pi}{4}$ ✓

$I_3 = \frac{1}{4}\left[2^{-2} + 3I_2\right]$

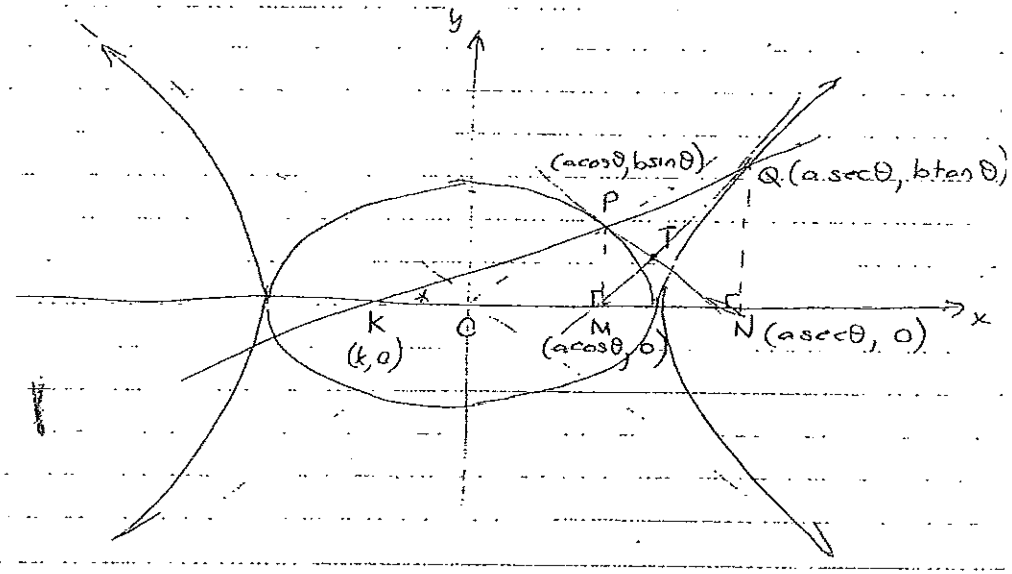
$= \frac{1}{16} + \frac{3}{4}\left[\frac{1}{2}(2^{-1} + I_1)\right]$

$= \frac{1}{16} + \frac{3}{16} + \frac{3}{8} \times \frac{\pi}{4}$

$= \frac{8+3\pi}{32}$ ✓

b)

NTS



b) i) Clearly $\Delta KPM \,|||\, \Delta KQN$ (equiangular)

So $\frac{KM}{KN} = \frac{PM}{QN}$ (matching side in similar Δ's)

$= \frac{b\sin\theta}{b\tan\theta}$ ✓

$= \cos\theta$

ii) Let $K = (k, 0)$, then:

$KM = KN\cos\theta$

$a\cos\theta - k = (a\sec\theta - k)\cos\theta$ ✓

$a\cos\theta - k = a - k\cos\theta$

$a(\cos\theta - 1) = k(1 - \cos\theta)$

$\therefore k = -a$

$\therefore K(-a, 0)$ ✓

iii) Gradient of tangent at P:

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= \frac{b\cos\theta}{-a\sin\theta}$ ✓

Point-gradient formula:

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$\frac{bx\cos\theta}{ab} + \frac{ay\sin\theta}{ab} = \frac{ab}{ab}(\cancel{\cos^2\theta + \sin^2\theta})$ ✓

$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

When $y = 0$:

$\frac{x\cos\theta}{a} = 1$

$\therefore x = a\sec\theta$ ✓

$\therefore$ the tangent passes through $N(a\sec\theta, 0)$

iv) α) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ — ①

$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ — ②

from given information

① ÷ $\cos\theta$: $\frac{x}{a} + \frac{y\tan\theta}{b} = \sec\theta$ — ①*

①* + ②: $\frac{x}{a}(1 + \sec\theta) = \sec\theta + 1$

$\frac{x}{a} = 1$

$x = a$

Sub into ①:

$\cos\theta + \frac{y\sin\theta}{b} = 1$

$y = \frac{b(1 - \cos\theta)}{\sin\theta}$

$\left(\text{or } \frac{b\sin\theta}{1 + \cos\theta}\right)$

$\therefore T\left(a, \frac{b(1 - \cos\theta)}{\sin\theta}\right)$

so T always lies on the line $x = a$. ✓

β) $y = \dfrac{b(1-\cos\theta)}{\sin\theta}$ Let $t = \tan\dfrac{\theta}{2}$

$\therefore \sin\theta = \dfrac{2t}{1+t^2}$

$\cos\theta = \dfrac{1-t^2}{1+t^2}$

$= \dfrac{b\left(1 - \frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}}$

$= \dfrac{b\left(1+t^2-(1-t^2)\right)}{2t}$

$= \dfrac{b \times \cancel{2}t^{\cancel{2}}}{\cancel{2t}}$

$= bt$

$= b\tan\dfrac{\theta}{2}$

$-1 < \tan\dfrac{\theta}{2} < 1$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

$\therefore -b < b\tan\dfrac{\theta}{2} < b$

$\therefore$ T lies on the line $x = a$
where $-b < y < b$ ✓

BUT $y \neq 0$ since P & Q are distinct points $\therefore$ the range of y-values for T should exclude $y = 0$

γ) The algebra above does not change, so T is still on the line $x = a$

for $\dfrac{\pi}{2} < \theta < \pi$, $b < b\tan\dfrac{\theta}{2} < \infty$
& for $-\pi < \theta < -\dfrac{\pi}{2}$, $-\infty < b\tan\dfrac{\theta}{2} < -b$

$\therefore$ T lies on the line $x = a$
where $y < -b$ or $y > b$ ✓