Differentiation

Revision Exercise 7

1 Use the definition of $f'(x)$ as a limit to find the derivatives of the following functions:

(a) $f(x) = \sqrt{x}$ (c) $f(x) = \frac{1}{\sqrt{x}}$

(b) $f(x) = \sqrt{6x - 4}$ (d) $f(x) = 2x^2 + x + 4$

2 Differentiate with respect to x, and simplify your answer as far as possible:

(a) $e^{-2x}(3 \cos 3x + 2 \sin 2x)$

(b) $\sin^3 (x^2 + 2)$

(c) $(x^2 + 2x)e^{x^2 + 2x}$

(d) $\log_e (x^3 e^{-x})$

3 Differentiate each of the following with respect to x:

(a) $(x + 1)e^{2x^2}$ (c) $\sqrt{\sin x}$

(b) $\sin^2 (\sqrt{x})$ (d) $\frac{1 - x^2}{\sqrt{1 + 2x}}$

4 Find $\frac{dy}{dx}$ for each of the following:

(a) $xy^3 - 3x^2 - xy = 8$

(b) $e^{xy} + y \ln x = \cos 2x$

(c) $xy = \ln y + 2$

(d) $x^2 y + y^3 = 18$

5 If $xy - e^{2x - 2} = 3$, find $\frac{dy}{dx}$ in terms of x and y.

Given that $y = 4$ when $x = 1$, find an approximate small change in x when y increases from 4 to 4.1.

6 The coordinates (x, y) of a point on the curve are given in terms of a parameter t by

$$x = t - \frac{2}{t} \text{ and } y = t - \frac{1}{t}, \text{ for } t \neq 0.$$

(a) Find the equation of normal to the curve at the point where $t = 1$.

(b) Find the values of t for which both x and y have the same rate of change with respect to t.

7 Sketch the graphs of $y = \frac{x}{x^2 + 1}$ and $y = x^2 - 1$, clearly indicating any asymptote(s) and turning points.

Hence, or otherwise, determine the number of real roots of the equations $x^4 - x - 1 = 0$, and show algebraically that one of the root is inside the interva $[1, 1.3]$.

Use the Newton-Raphson method to find an approximate root inside the interval $[1, 1.3]$, correct to two decimal places.

8 Sketch on the same diagram the graphs of $y = x - 1$ and $y = ke^{-3x}$, where $-1 < k < 0$.

State the number of real roots of the equation $ke^{-3x} - x + 1 = 0$.

For the case $k = 1$, sketch appropriate graphs to show that the equation

$$e^{-3x} - x + 1 = 0$$

has exactly one real root.

Denoting this real root by α, find the integer n such that the interval $[n - 1, n]$ contains α.

Use the Newton-Raphson method to find the value of α correct to 3 decimal places.

9 Find, stating your reason, the value of the positive integer n such that

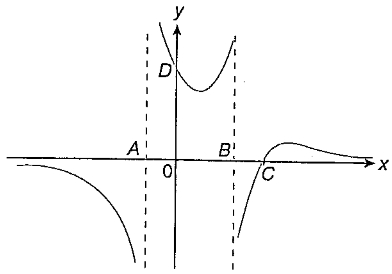
$$n - 1 \leqslant \sqrt[3]{100} \leqslant n$$

Hence use the Newton-Raphson method to find an approximate value of $\sqrt[3]{100}$ correct to 3 decimal places.

10 The parametric equations of a curve are $x = \ln \sin \theta$, $y = \ln \cos \theta$, $0 < \theta < \frac{\pi}{2}$.

Show that the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$ is given by the equation $x + y + \ln 2 = 0$.

11 The curve of $y = \frac{3x - 9}{x^2 - x - 2}$ is shown in the figure below. Find the coordinates of the point A, B, C and D.



By expressing y in partial fractions, obtain an expression for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(a) Show that $(5, \frac{1}{3})$ is a maximum point and find the coordinates of the minimum point.

(b) Show that the equation of the curve can be written in the form $yx^2 - (y + 3)x - (2y - 9) = 0$ and has no real roots if $(y - 3)(3y - 1) < 0$. Solve this inequality and comment on your answer.

9 Since $64 \leqslant 100 \leqslant 125$
i.e. $4 \leqslant \sqrt[3]{100} \leqslant 5$
$\therefore n = 5$
$\sqrt[3]{100} \simeq 4.642$

11 $A(-1, 0), B(2, 0), C(3, 0), D(0, \frac{9}{2})$

$y = \frac{4}{x+1} = \frac{1}{x-2}$

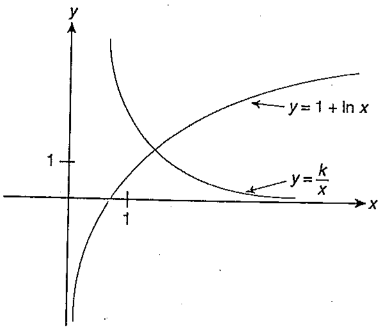
$\frac{dy}{dx} = \frac{-4}{(x+1)^2} + \frac{1}{(x-2)^2}$

$\frac{d^2y}{dx^2} = \frac{8}{(x+1)^3} - \frac{2}{(x-2)^3}$

(a) minimum point (1, 3)

(b) The curve of y is not defined in the region $\{y \mid \frac{1}{3} < y < 3\}$

12



For all $k > 0$, the graphs $y = 1 + \ln x$ and $y = \frac{k}{x}$ intersect at only 1 point so the equation $f(x) = 0$ has exactly one real root for all $k > 0$.
$1 < k < 2 + 2\ln 2$
1.445

13 0.0544

14 (a) $V = \pi x^2 h = \pi x^2(6 - 3x) = 6\pi x^2 - 3\pi x^3$
(b) $\frac{32\pi}{9}$ cm^3

15 $h = \frac{100}{\pi r^2}$, –1%

16 $\frac{dy}{dx} = \frac{1}{\sin 2t} = \text{cosec } 2t$
$6y + 3\sqrt{3}\,x = 21 - \sqrt{3}\,\pi$

17 height of trapezium $h = \sqrt{16 - x^2}$
$A = \frac{1}{2}(2x + 8)\sqrt{16 - x^2}$

$= (4 + x)^{\frac{3}{2}}(4 - x)^{\frac{1}{2}}$

(a) $\frac{dA}{dx} = (4 - 2x)\sqrt{\frac{4+x}{4-x}}$

18 Let r = radius of the sphere
Height of the cone, $h = \frac{r}{\sin\theta} + r = \frac{r(1 + \sin\theta)}{\sin\theta}$

Radius of the cone, $R = h \tan c = \frac{r(1 + \sin\theta)}{\cos\theta}$

Volume of the sphere, $V = \frac{4}{3}\pi r^3$

Volume of the cone, $V_c = \frac{1}{3}\pi R^2 h$

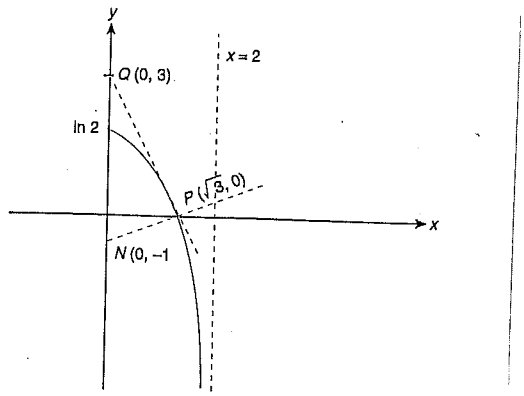
$\frac{dV_c}{dx} = \frac{V(1 + x)(3x - 1)}{4[(1 - x)x]^2}$

Volume of cone is maximum when $x = \frac{1}{3}$, and $V_c = 2V$

19 $r = \left(\frac{3k}{4\pi}\right)^{\frac{1}{3}} D^{-\frac{1}{3}}$

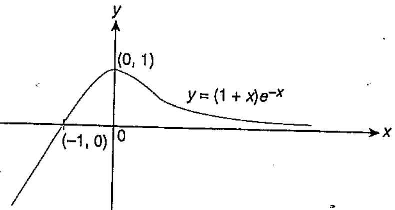
$\frac{dr}{dt} = \left(\frac{3k}{4\pi}\right)^{\frac{1}{3}}\left(-\frac{1}{3}D^{-\frac{4}{3}}\right)\frac{dD}{dt}$
$= \sqrt[3]{\frac{4k}{9\pi h^4}}$

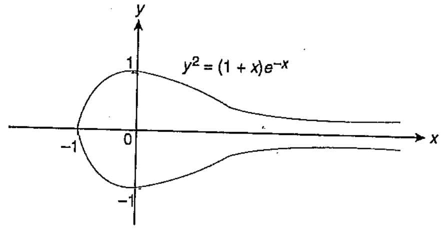
20 (a) $P(\sqrt{3}, 0)$
(b) $\frac{dy}{dx} = -\frac{\sqrt{4 - t^2}}{t^2} \leqslant 0$ for 0 for $0 < t \leqslant 2$



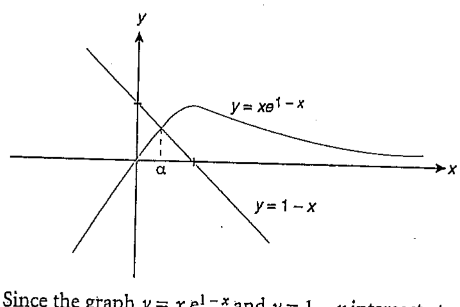
(c) tangent: $y + \sqrt{3}x = 3$
normal: $\sqrt{3}y = x - \sqrt{3}$
Area of $\Delta PQN = \frac{1}{2}(4)(\sqrt{3}) = 2\sqrt{3}$

21





$y = xe^{1-x} = [1 + (x - 1)]e^{-(x-1)}$
Transformation is a translation of 1 unit along the positive x-axis.



Since the graph $y = x\,e^{1-x}$ and $y = 1 - x$ intersect at exactly one point, hence $x(1 + e^{1-x}) = 1$ has exactly one root α.
$\alpha = 0.341$

22 $\tan\theta = \frac{x}{10}$
$\frac{d\theta}{dt} = -\frac{3}{20}\pi$/s

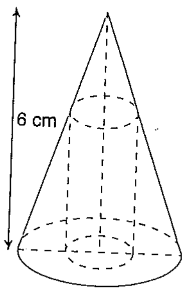
23 $P(4m, 2m^2)$
$N(0, 2m^2 + 4)$
Percentage change in A $= \frac{3m^2 + 1}{m^2 + 1}$
When m gets very large, percentage change in A tends to 3%

24 (a) (i) $x(\ln 3x)^{x^2}\left[2\ln(\ln 3x) + \frac{1}{\ln 3x}\right]$
(ii) $2e^{-2x}\tan 5x\,[5\sec^2 5x - \tan 5x]$

12 Given that $f(x) = \ln x - \frac{k}{x} + 1$ where $x > 0$ and k is a real positive constant. Show, by sketching two appropriate curves on the same diagram, that the equation $f(x) = 0$ has exactly one real root for all $k > 0$.
Show that f is an increasing function for $x > 0$, and hence find the range of values of k such that the equation $f(x) = 0$ has one real root α in the interval $(1, 2)$.
If $k = 2$ and α lies in the interval $(1, 2)$; obtain, by the Newton-Raphson method, an approximation of α, giving 3 decimal places in your answer.

13 Given that $y = x \ln x$, find the approximate change in y due to a 1% increase in x when $x = e$. Give your answer correct to 4 decimal places.

14



The figure above shows a right circular cone of base radius 2 cm and height 6 cm standing on a horizontal table.
A cylinder of radius x cm stands inside the cone with its axis coincident with the axis of symmetry of the cone and such that the cylinder touches the curved surface of the cone as shown. The volume of the cylinder is V cm^3.

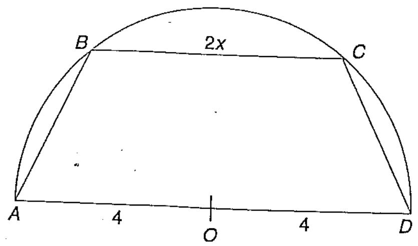
(a) Show that $V = 6\pi x^2 - 3\pi x^3$

(b) Given that x can vary, obtain the maximum value of V.

15 A cylinder with radius r and height h has a fixed volume of 100 cm^3. Find the relationship between r and h. Hence, find the percentage change in its height when its radius is increased by 0.5 percent.

16 A curve has parametric equations $x = \tan 2t - 2t$, $y = \sec 2t$, $(0 < t < \frac{\pi}{4})$. Find $\frac{dy}{dx}$, giving your answer in the form of a single trigonometric function. Find the equation of the normal at the point when $t = \frac{\pi}{6}$.

17

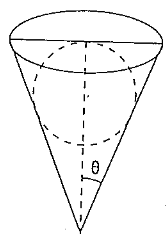


The diagram above shows a trapezium $ABCD$ inscribed in a semicircle with radius 4 cm. If $BC = 2x$ cm, show that A, the area of trapezium $ABCD$ is $(4 + x)^{\frac{3}{2}} (4 - x)^{\frac{1}{2}}$ cm^2.

(a) As x varies, find $\frac{dA}{dx}$.

(b) Show that A is maximum when $x = 2$ cm.

18 A designer recently bought a crystal sphere with a constant volume V. To display this item among his collection, he wishes to design a glass container in the shape of a right circular cone such that the cone is circumscribed about the sphere as shown on the right. The semi-vertical angle of the cone is θ.



Show that the volume of the cone is given by

$$\frac{V(1 + x)^2}{4x(1 - x)}$$

where $x = \sin \theta$.
Determine the minimum volume the cone may have in terms of V.

19 A spherical bubble of radius r cm is formed at the base of a tank containing water of height h cm. The depth of the bubble inside the water is defined as the vertical distance from the water surface to the centre of the bubble.
At time t seconds after the formation of the bubble, the depth and volume of the bubble is given by D cm and V cm^3 respectively. The volume of the bubble is inversely proportional to its depth at any time, i.e. $V = \frac{k}{D}$, where k is a constant.
Given that the bubble is moving vertically upwards at a constant rate of 1 cm s^{-1}, find the rate at which its radius is changing at the instant when the bubble is half way to the water surface. Leave your answer in terms of k and h.

20 A curve is given by the parametric equations:

$$x = \sqrt{4 - t^2}, y = \ln t, 0 < t \leqslant 2.$$

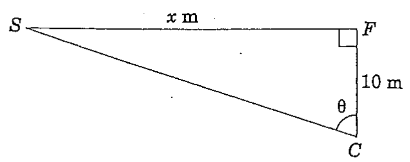
(a) Find the coordinates of P, the point at which the curve cuts the x-axis.

(b) Show that $\frac{dy}{dx} \leqslant 0$ for $0 < t \leqslant 2$ and hence sketch the graph of the curve and show clearly any asymptotes.

(c) The tangent at P cuts the y-axis at Q and the normal at P cuts the y-axis at N.
Find the area of the triangle PQN.

21 Sketch the graph of $y = (1 + x)e^{-x}$, indicating clearly the turning points and asymptotes if any. Hence, on a separate diagram, sketch the graph of $y^2 = (1 + x)e^{-x}$.
State the transformation by which the graph of $y = xe^{1-x}$ may be obtained from the graph of $y = (1 + x)e^{-x}$.
By means of a suitable sketch, deduce that $x(1 + e^{1-x}) = 1$ has exactly one real root α.
Show that α lies between 0.3 and 0.4.
Use the Newton-Raphson method to obtain an approximation for α correct to 3 significant figures.

22



A television crew is televising a 100 m sprint event. The cameraman positions himself 10 m from F, a point on the finishing line.
The point S represents the location of the first athlete to come in. He is running towards F such that angle $SFC = \frac{\pi}{2}$. At time t seconds, $SF = x$ m and angle $FCS = \theta$ radians. When the athlete is 10 m away from F, he is running at 3π m s^{-1}. Find the rate of change of θ at this moment.

23 A curve has the equation of $8y = x^2$. Given that the line $y = mx - km^2$ is a tangent to this curve for all real values of m, show that $k = 2$.
Find the coordinates of the point P (in terms of m) at which the tangent meets the curve and the coordinates of the point N where the normal to the curve at the point P cuts the y-axis.
If T is the point where the tangent cuts the y-axis, show that the area, A, of triangle PNT is given by $A = 8m(m^2 + 1)$.
Using the method of small increments, find, in terms of m, the approximate percentage change in the value of A when m increases by 1%. Comment on the percentage change in A when m gets very large.

24 (a) Differentiate the following with respect to x:
(i) $(\ln 3x)^{x^2}$
(ii) $e^{-2x}\tan^2 5x$
(b) Show that the normal to the curve $y = \tan x$ at the point P whose coordinates are $(\frac{\pi}{4}, 1)$ meets the x-axis at the point $Q\,(\frac{\pi + 8}{4}, 0)$.

Revision Exercise 7

1 (a) $f'(x) = \frac{1}{2\sqrt{x}}$ (c) $f'(x) = \frac{1}{2\sqrt{x^3}}$
(b) $f'(x) = \frac{3}{\sqrt{6x-4}}$ (d) $f'(x) = 4x + 1$

2 (a) $-13e^{-2x}\sin 3x$
(b) $6x\sin^2(x^2+2)\cos(x^2+2)$
(c) $2(x+1)^3 e^{x^2+2x}$
(d) $\frac{1}{x}(3-x)$

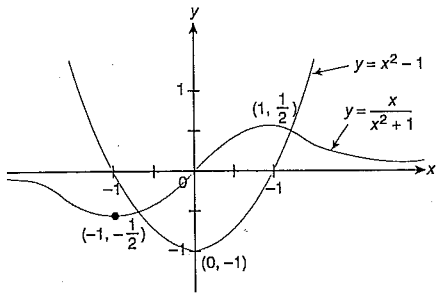
3 (a) $(2x+1)^2 e^{2x^2}$ (c) $\frac{\cos x}{2\sqrt{\sin x}}$
(b) $\frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}} = \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$ (d) $\frac{-(3x^2+2x+1)}{(1+2x)^{\frac{3}{2}}}$

4 (a) $\frac{dy}{dx} = \frac{6x - y^3 + y}{3xy^2 - x}$ (c) $\frac{dy}{dx} = \frac{y^2}{(1-xy)}$
(b) $\frac{dy}{dx} = \frac{2x\sin 2x + xye^{xy} + y}{x^2e^{xy} + x\ln x}$ (d) $\frac{dy}{dx} = \frac{-2xy}{x^2 + 3y^2}$

5 $\frac{dc}{dx} = \frac{2e^{2x-2} - y}{x}$
-0.05

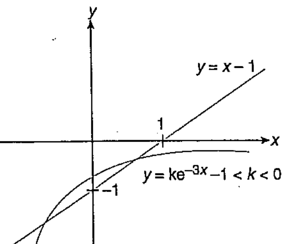
6 $\frac{dc}{dx} = \frac{1+t^2}{2+t^2}$
(a) $2y + 3x + 3 = 0$
(b) There is no value for t at which both x and y have the same rate of change with respect to t.

7

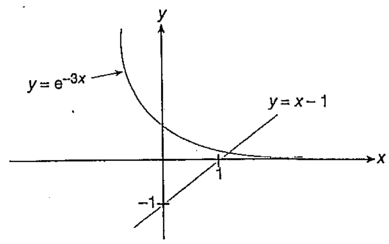


asymptote $y = 0$
turning point of $y = \frac{x}{x^2+1}$: $(1, \frac{1}{2})$ maximum
$(-1, -\frac{1}{2})$ minimum
turning point of $y = x^2 - 1$: $(0, -1)$ minimum
$x^4 - x - 1 = 0$
$x^4 - 1 = x$
$\therefore (x^2 - 1)(x^2 + 1) = x$
$\therefore x^2 - 1 = \frac{x}{x^2+1}$
Number of roots = 2
Root = 1.22

8



Number of roots is 2.



Since there is only one point of intersection, and so the number of roots is only 1.
n = 2
$\alpha = 1.044$