Rates of change

Quick Review 7.6 (e)

1 A population of bacteria is growing according to the law

$$N(t) = 0.01e^t + 10t + 1000$$

where t is measured in hours. What is the rate of change of population with respect to time when $t = 10$?

2 A population of a certain species of birds in a national forest reserve follows the law

$$N(t) = \frac{1000e^t}{1 + e^t}$$

where t is in years. Find the rate of change of N with respect to t.

3 The amount of money a family saves is a function $S(x)$ of its income x. The marginal propensity to save is $\frac{dS}{dx}$. If

$$S(x) = \frac{2x^2}{5(50000 + x)}$$

Find the marginal propensity to save.

4 The cost function for a certain type of wild catfish was estimated to be

$$C(x) = \frac{7.12}{13.74x - x^2}$$

where x is the number of millions of kilograms of catfish caught and $C(x)$ is the cost in million of ringgit. Find the marginal cost.

5 The demand function (i.e. selling price function) for a certain product is given by

$$p(x) = \frac{10}{\sqrt{1 + x}}$$

Find the rate of change of price p with respect to the number of units sold x; and also estimate the approximate change in price p as x changes from 99 to 102.

6 The cost equation for a certain product is given by

$$C(x) = (x + 100)\sqrt{x^3} + 1000$$

where x is the number of units produced. Find the rate of change of cost C with respect to x; and also estimate the change in costs C as x changes from 900 to 904.

7 The surface area S of a sphere of radius r is given by the formula $S = 4\pi r^2$.

(a) How fast is the surface area changing with respect to the radius when the radius is 4 cm?

(b) What is the change in surface area when the radius decreases from 4 cm to 3.95 cm?

(c) A 2% error is made in measuring the radius of the sphere. What will be the percentage error in the surface area?

(d) If the volume of the sphere increases by 2%, find the corresponding percentage increase in surface area.

8 The volume of water in a spherical vessel of depth h cm is given by the expression

$$V = \frac{\pi h^2}{3}(15 - h)$$

(a) Find the rate of change of volume with respect to h.

(b) Find the amount of water which must be poured into the vessel to increase its depth from 4 cm to 4.1 cm.

9 The lengths of two sides of a triangle are measured to be exactly 8 cm and 10 cm. The included angle θ radian is measured to be 45° with an error of 0.5°.
Find the rate of change of the area with respect to θ, and calculate the approximate percentage error in the calculation of the area of the triangle.

10 The height of a lamp post is estimated by measuring the length of the shadow cast by a 2 m long vertical pole placed 10 m away from the base of the lamp post.
The lamp is at the top of the post, and the shadow was measured to be 2.00 m with an error of 1 cm. Use a derivative to estimate the error in measuring the height of the lamp post.

11 Use derivatives to estimate the following quantities:

(a) $\sqrt{627}$ (d) $\sqrt{4.016} + \frac{1}{\sqrt{4.016}}$

(b) $\sqrt{9.01}$ (e) $\sqrt[3]{0.065}$

(c) $\sqrt[3]{(8.35)^2}$ (f) $\sqrt[3]{(7.7)^2}$

12 A particle moves in a straight line and its distance (S ft) from the point at which it is situated at zero time is given in terms of the time (t sec) by the formula

$$S = 45t + 11t^2 - t^3$$

Find the velocity and acceleration after 3 sec, and prove that the particle will come to rest after 9 sec.

13 A particle moves along a straight line in such a way that its distance S ft from the starting point P after t sec is given by the formula:

$$S = 27t - 2t^2$$

What are its velocity and acceleration after $6\frac{3}{4}$ sec? How long does it take for the velocity to be reduced from 15ft/sec to 9ft/sec and how far does the particle travel meanwhile?

14 A stone is thrown vertically upwards at 96 ft/sec from a point on the level with but just beyond a cliff ledge. Its height h ft above the ledge t sec later is

$$h = 16t(6 - t)$$

If its velocity at any instant t is V ft/sec.

(a) Find the velocity V ft/sec.

(b) When is the stone at the ledge level?

(c) When is the stone momentarily at rest? What is the greatest height reached?

(d) Find the total distance travelled during the 3rd and 4th seconds.

15 The cost of producing x units of a particular product is given by

$C(x) = x^3 - 30x^2 + 320x + 1000$ (ringgit)

If the demand function (i.e. selling price function) is $p(x) = 500 - 0.5x$, where p is the selling price per unit. Determine.

(a) an expression for the total revenue function, $R(x)$.

(b) an expression for the total profit function, $P(x)$.

(c) Find expressions for the marginal revenue, $R'(x)$, the marginal cost, $C'(x)$ and the marginal profit.

16 A charitable organisation is arranging an evening of music and performance with a light meal supplied. They feel that they can sell x tickets if they set the price per ticket at $p(x) = 15 - 0.01x$ ringgit.

If x tickets are sold their costs for running the evening will be RMC where

$$C(x) = 700 + 3x$$

that is, RM700 is the fixed cost and the cost per ticket sold is RM3.

(a) Find an expression for the total revenue $R(x)$ received if x tickets are sold.

(b) Find an expression for $P(x)$, the total profit if x tickets are sold.

(c) Determine expressions for marginal revenue, $R'(x)$; marginal cost, $C'(x)$; and marginal profit, $P'(x)$.

(d) Determine the marginal revenue, marginal cost and marginal profit for $x = 200$.

17 The cost and revenue function (in ringgit) of producing and selling x units of a product is given by $C(x) = 3.20 + 1.40x$ and $R(x) = 0.1x^2 - 0.001x^3$ respectively.

Find the marginal cost and marginal profit for producing and selling x units.

Find the value of x that will maximise the profit and marginal profit.

18 A 7 meter ladder leans against a vertical wall. The bottom of the ladder is slipping away from the base of the wall at the rate of 0.1 meter per second. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 2 meter from the base?

19 A ship A is moving eastward toward a fixed point P at a speed of 12 knots (nautical miles per hour). At the moment when ship A is 72 nautical miles from P, another ship B passes through P, heading due north at 10 knots. How fast is the distance between the ships changing two hours after ship B has passed through P?

20 A boat passes a fixed buoy at 7 a.m. heading due east at 3 km per hour. Another boat passes the same buoy at 8 a.m. heading due south at 5 km per hour. How fast is the distance between the boats changing at 9.30 a.m.?

21 A boy is flying a kite, which is at a height of 112 feet. The wind is carrying the kite horizontally away from the boy at a speed of 6 feet per second. How fast must the string be let out when the kite is 130 feet away from the boy?

22 Water is pouring into an inverted right circular cone at the rate of 0.1 cubic meters per minute. The height of the cone is 10 meters and the radius of its base is 1 meter. Find the rate at which the depth and the surface area of the water is increasing when the water is half-way up the cone.

Quick Review 7.6 (e)

1 230.26

2 $\frac{1000e^t}{(1+e^t)^2}$

3 $\frac{2(100000x+x^2)}{5(50000+x)^2}$

4 $\frac{7.12(2x-13.74)}{(13.74x-x^2)^2}$

5 $\frac{dp}{dx} = -\frac{5}{(1+x)^{\frac{3}{2}}}$, -0.015

6 $\frac{dc}{dx} = \frac{5}{2}x^{\frac{3}{2}} + 150x^{\frac{1}{2}}$, 288000

7 (a) 32π (c) 4%

(b) 1.6π cm^2 (d) $1\frac{1}{3}$%

8 (a) $\frac{dV}{dh} = \pi h(10-h)$

(b) 2.4π cu. cm

9 $\frac{dA}{d\theta} = 40\cos\theta$, $\frac{5\pi}{18}$%

10 0.05 m

11 (a) $\frac{626}{25} = 25.04$

(b) $\frac{1801}{600} \approx 3.00167$

(c) $\frac{12.35}{3} = 4.1167$

(d) 2.503

(e) $\frac{193}{480} \approx 0.4021$

(f) 3.9

12 84 ft/sec, 4 ft/sec^2

13 0 ft/sec, -4 ft/sec, $1\frac{1}{2}$ sec, 18 ftt

14 (a) $V = 96 - 32t$

(b) $t = 0$, 6 sec

(c) $t = 3$, 144 ft

(d) 32 ft

15 (a) $R(x) = 500x - 0.5x^2$

(b) $P(x) = -1000 + 180x + 29.5x^2 - x^3$

(c) $R'(x) = 500 - x$

$C'(x) = 3x^2 - 60x + 320$

$P'(x) = 180 + 59x - 3x^2$

16 (a) $R(x) = 15x - 0.01x^2$

(b) $P(x) = -700 + 12x - 0.01x^2$

(c) $R'(x) = 15 - 0.02x$

$C'(x) = 3$

$P'(x) = 12 - 0.02x$

(d) $R'(200) = 11$

$C'(200) = 3$

$P'(200) = 8$

17 RM1.40; $0.2x - 0.003x^2 - 1.40$

59, 33

18 $\frac{\sqrt{5}}{75}$ ms^{-1}

19 $-\frac{376}{52}$ knots

20 $4\sqrt{2} \approx 5.64$ km/hr

21 $\frac{198}{65} \approx 3.04$ ft/sec

22 $\frac{2}{5\pi}$ m/min, 0.04 sq. m./min