Prediction and representation of array performance under sensor failure

Erdal Mehmetcik[1], Çağatay Candan[2]

[1]Underwater Acoustic Systems, Defense Systems Business Sector, ASELSAN Inc.
[2]Electrical-Electronics Engineering Department, Middle East Technical University

Abstract — Military sensor arrays are exposed to harsh conditions in general. In certain cases, the operator or the system may be required to disable certain elements of the array due to their unreliable readings. The system performance is affected with the number of the disabled sensors and their locations in the array. In some cases, even if the majority of the sensors are disabled, the system can be still useful for direction finding in a particular angular sector. In such cases, the operator needs a performance prediction tool to assess the reliability of array with disabled sensors. As the number of possible combinations for disabled sensors can be very large, it is more practical to assess the array performance with an online tool. Our goal is to provide the operator, a prediction for the system performance with a set of disabled sensors, in a simple and concise manner. Since the performance deviation depends on the signal to noise ratio, bearing angle and the operating frequency, the representation of 4-dimensional data is also a challenge. In this paper, we utilize an approximate deterministic Ziv-Zakai type lower bound as a means for the array performance prediction. Also we present a visualization scheme for the display of performance metrics, for instance direction finding uncertainties, which depend on frequency, bearing angle and signal to noise ratio.

1 Introduction

Military sensor arrays are deployed in harsh conditions in general. Although the array and its elements are designed to withstand the expected harsh conditions, it is always possible to have sensor failures. It is possible that due to an accident, some elements of the array can be damaged, i.e., not able to function properly. In such cases, the operator or the system may be required decide to disable some of the sensors with unreliable readings. The system performance is affected by the number of the disabled sensors, and their location in the array. In some cases, even if majority of the sensors are disabled, the overall system can still be useful for direction finding purposes in a particular angular sector. In such cases, the operator needs a performance prediction tool to assess the reliability of array with disabled sensors. As the number of possible combinations for the disabled sensors can be very large, it is more practical to assess the array performance with an online tool rather than reporting the performance on a case-by-case manner typically in a user manual. Our problem is to provide the operator, a prediction on the array performance in a rather simple and concise manner. Since the performance deviation depends on the signal to noise ratio, bearing angle and the operating frequency; a representation for the illustration of 4-dimensional data is also required.

Performance bounds can be divided into two main classes, namely deterministic bounds (for non-random parameter estimation) and Bayesian bounds (for random parameter estimation). Traditionally the most popular method to assess array performance is the Cramer-Rao Bound (CRB) [1]. CRB is easy to compute, has both deterministic and Bayesian versions, and it is independent of the estimator being used. For direction finding (DF) applications, CRB is basically dependent on the second derivative of the mainlobe of the beampattern of the array for the Gaussian case. Under "low" signal to noise ratio (SNR) conditions, the maximum likelihood estimate can be around one of the side-lobes of the beampattern. CRB provides an optimistic lower bound for such cases. The SNR value below which the performance of an efficient estimator starts to deteriorate is called the threshold SNR value. CRB is not a tight bound below the threshold SNR value. There is a vast literature on this topic (see for instance [2]) and many different lower bounds are proposed to account for the threshold region performance, for instance Barankin Bound [3], Weiss Weinstein Bound [4] and Ziv-Zakai Bound [5], [6]. These Bayesian bounds are able to model the threshold region performance, though they are somewhat tedious to calculate. In case of sensor failures, we suggest to use one of these bounds to observe the change in the threshold SNR.

In this paper, we derive an approximate deterministic Ziv-Zakai type bound for performance assessment of arrays with faulty sensors.

2 Problem definition

We consider the following signal model;

$$r_n = \eta_n B_n A_s e^{j\psi_n} + \omega_n, \; A_s \in \mathbb{R} \tag{1}$$

$$\boldsymbol{r} = \begin{bmatrix} r_0 \\ r_0 \\ \vdots \\ r_{N-1} \end{bmatrix}, \; \boldsymbol{a}_\psi = \begin{bmatrix} B_0 \\ B_1 e^{j\psi_1} \\ \vdots \\ B_{N-1} e^{j\psi_{N-1}} \end{bmatrix} \tag{2}$$

$$\eta_n \sim \mathrm{CN}(\mu_\eta, 2\sigma_\eta^2) \tag{3}$$
$$\omega_n \sim \mathrm{CN}(0, 2\sigma_n^2) \tag{4}$$

where A_s is the signal amplitude, B_n is the sensor response at the direction of arrival (DOA), η_n represents the signal fading coefficient, $CN(\mu, \sigma^2)$ represents a complex Gaussian distribution with mean μ and variance σ^2 and $(.)^H$ is the Hermitian transpose. ψ_n is the phase value associated with the relative delay of the n^{th} sensor with respect to the first sensor. Consequently the signal mean vector and covariance matrix are as follows,

$$\boldsymbol{\mu} = E\{\boldsymbol{r}\} = \mu_\eta A_s \boldsymbol{a}_\psi \tag{5}$$

$$\begin{aligned}\boldsymbol{C}_x &= E\{(\boldsymbol{r}-\boldsymbol{\mu})(\boldsymbol{r}-\boldsymbol{\mu})^H\} \\ &= 2\sigma_\eta^2 A_s^2 \boldsymbol{a}_\psi \boldsymbol{a}_\psi^H + 2\sigma_\omega^2 \boldsymbol{I}_N\end{aligned} \tag{6}$$

where $E\{\cdot\}$ is the expectation operator and $\boldsymbol{I}_N$ is an N by N identity matrix.
In the direction finding applications we have the following form for ψ_n,

$$\psi_n = \frac{2\pi}{\lambda}(\mathbf{p}_n^H - \mathbf{p}_0^H)\,\mathbf{u} \tag{7}$$

$$\mathbf{p}_n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{8}$$

$$\mathbf{u} = \begin{bmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{bmatrix} \tag{9}$$

where λ is the wavelength, $\mathbf{p}_n$ is the position vector of the n^{th} sensor and $\mathbf{u}$ is the unit vector pointing at direction of arrival defined by azimuth and elevation angles ϕ and θ.
Especially in shallow waters, multipath signal propagation is an important problem. Consequently the complex fading coefficient η_n is added to the signal model. We assume Rician fading conditions, and we define

$$K \triangleq \frac{|E\{\mu_\eta\}|^2}{E\{|\eta-\mu_\eta|^2\}} = \frac{|\mu_\eta|^2}{2\sigma_\eta^2} \tag{10}$$

as the Rician factor. The Rician factor is a descriptor of the ratio between the power of the signal component that arrives the sensor array through the direct path, and the total power of the signal arriving through the reflections (multi-path arrivals). Consequently, when the Rician factor is equal to zero, we do not observe the direct path signal. At the other extreme when the Rician factor tends to infinity, we only observe the direct path arrival and do not have any multi-path arrivals.

Another definition we need is the gross error. At high signal to nose ratio (SNR) values, the estimates will have a small jitter inside the main-lobe of the beampattern and the error here is generally called "fine error". However, when the input SNR drops below a certain threshold value, the estimates start to switch back and forth between main-lobe and side-lobes. Consequently causing a large ("gross") error. This situation is illustrated in Fig. 1 where a maximum likelihood estimator is used with a sparse array (to emphasize the high side-lobes) just below the SNR threshold for this array configuration. As seen in the figure, the likelihood of the estimates roughly follow the shape of the beampattern. We will later show that the SNR threshold value for which the estimator output remains within the main-lobe of the array can be found using the Ziv-Zakai bound.

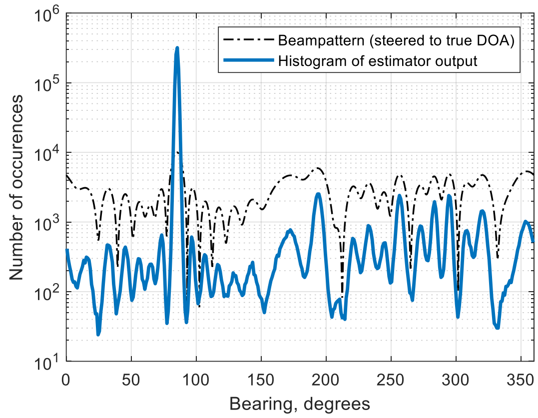

Fig. 1. Histogram of Maximum Likelihood Estimator outputs for 10^6 Monte Carlo runs with the true DOA at 85.3 degrees.

Using this signal model, the bearing estimation performance can be studied by the performance bounds (Cramer Rao, Barankin, Ziv-Zakai etc.) that provide a lower bound for the estimator accuracy. We will utilize Ziv-Zakai bounds to estimate the system performance along with the threshold SNR estimation in the latter sections. But first, we briefly examine the Cramer Rao lower bound which is a fundamentally important bound providing a tight characterization of the maximum likelihood estimator at high SNR conditions. We also use the CRB as a comparison metric with other bounds.

3 Performance Bounds

3.1. Cramer-Rao Lower Bound

Using the derivation for complex Gaussian case in [7] (see Appendix 15C) with a fading model, the Cramer-Rao Bound (CRB) is expressed as follows;

$$\begin{aligned}[\mathrm{FIM}]_{ij} = \mathrm{tr}&\left\{\mathbf{C}_x^{-1}(\xi)\frac{\partial \mathbf{C}_x(\xi)}{\partial \xi_i}\mathbf{C}_x^{-1}(\xi)\frac{\partial \mathbf{C}_x(\xi)}{\partial \xi_j}\right\} \\ &+ 2\mathrm{Re}\left\{\frac{\partial \boldsymbol{\mu}^H(\xi)}{\partial \xi_i}\mathbf{C}_x^{-1}(\xi)\frac{\partial \boldsymbol{\mu}(\xi)}{\partial \xi_j}\right\}\end{aligned} \tag{11}$$

$$\mathrm{CRB} = \mathrm{FIM}^{-1} \tag{12}$$

where $\mathrm{tr}\{\cdot\}$ is the trace operator, FIM is the Fisher Information Matrix. Although it seems tedious to calculate the expression analytically, it is indeed quite easy to calculate numerically. However, CRB provides extremely optimistic lower bounds at low signal to noise ratios. And it is quite insensitive to the channel fading effects. We will elaborate on these drawbacks in the following sections.

3.2 Ziv-Zakai Bound

The Ziv-Zakai Bound [2], [5], [6], [8] is defined as follows;

$$\text{ZZB} = \int_0^\infty \nu\{A(h)P_e(h)\}h\,dh \tag{13}$$

$$A(h) = \int_{-\infty}^{\infty} \min(p_u(u), p_u(u+h))\,du \tag{14}$$

where, $p_u(\cdot)$ is the prior distribution of the parameter to be estimated. $P_e(h)$ is the minimum probability of error for deciding between the true parameter θ and its offset version, $\theta + h$. And $\nu\{\cdot\}$ is defined as the valley-filing operator,

$$\nu\{f(h)\} = \max_{\epsilon \ge 0} f(h+\epsilon) \tag{15}$$

which returns a non-increasing output as seen in Fig. 2. For bearing estimation problem under consideration, the ZZB equations simplify [8] to the following set of equations,

$$u \sim unif(0, 2\pi) \rightarrow A(h) = \frac{2\pi - h}{2\pi} \tag{16}$$

$$\text{ZZB} = \int_0^\pi \nu\left\{P_e(h)\frac{2\pi - h}{2\pi}\right\}h\,dh \tag{17}$$

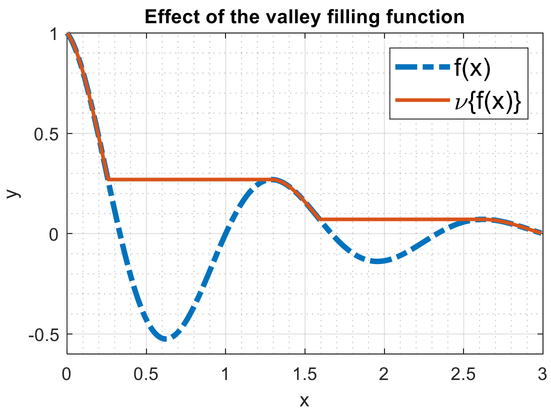

Fig. 2. The valley filling function.

The Ziv-Zakai bound is a Bayesian bound which requires a prior distribution on the DOA. Hence, different from a non-random parameter estimation bound, the Ziv-Zakai bound depends on the prior distribution but not on the actual value of the DOA. By using Bayesian bounds with the uniform pdf over an angular sector of interest, one does not get informed about a particular DOA, but receives only partial information about the average performance in that sector. To study the performance in different angular sectors, one can use different prior distributions.

Here we follow an alternative approach. We would like to obtain a performance bound specific for a fixed DOA, i.e. a non-random parameter bound. To do that, we express the event of $\{|\hat{\theta} - \theta_t| > \epsilon\}$ as the union of disjoint events $\{\hat{\theta} - \theta_t > \epsilon\}$ and $\{\hat{\theta} - \theta_t < -\epsilon\}$ for $\epsilon > 0$, where θ_t is the true target angle (non-random) and $\hat{\theta}$ is an estimator of θ_t. Consequently,

$$MSE_{\theta_t} = 2\int_0^{T_2 - T_1} \epsilon\, P\{|\theta_t - \hat{\theta}| > \epsilon\}\,d\epsilon \tag{18}$$

At this point we can obtain an approximate lower bound for the maximum likelihood (ML) estimator, by lower bounding the probability of $P\{|\theta_t - \hat{\theta}| > \epsilon\}$ given in (18). By quantizing the probability of error $P\{|\theta_t - \hat{\theta}| > \epsilon\}$ to $P_e\{\theta_t, \theta_t \pm \epsilon\}$, we get

$$\begin{aligned} MSE_{\theta_t}^{ML} \ge\, & 2\int_0^{\frac{T_2 - \theta_t}{2}} \epsilon\, P_e\{\theta_t, \theta_t + 2\epsilon\}\,d\epsilon \\ & +2\int_0^{\frac{\theta_t - T_1}{2}} \epsilon\, P_e\{\theta_t, \theta_t - 2\epsilon\}\,d\epsilon \end{aligned} \tag{19}$$

where $\theta_t \in [T_1, T_2]$ is the interval that θ_t lies. For the problem of interest $\theta_t \in [0, 2\pi)$. By setting $T_1 = \theta_t - \pi$ and $T_2 = \theta_t + \pi$, we have;

$$\begin{aligned} MSE_{\theta_t}^{ML} \ge\, & 2\int_0^{\frac{\pi}{2}} \epsilon\, P_e\{\theta_t, \theta_t + 2\epsilon\}\,d\epsilon \\ & +2\int_0^{\frac{\pi}{2}} \epsilon\, P_e\{\theta_t, \theta_t - 2\epsilon\}\,d\epsilon \end{aligned} \tag{20}$$

We will refer the last bound as the approximate deterministic ZZB from this point on. We use such a name for the bound in (20), since the bound can be converted to the conventional Ziv-Zakai bound when θ_t is assumed to be a random parameter. The term $P_e\{\theta_t, \theta_k\}$ refers to the probability of error between deciding θ_k given the true parameter is θ_t.

The problem now reduces to providing a closed form expression for the minimum probability of error between deciding $H_0: \psi = \psi_0$ and $H_1: \psi = \psi_0 + h$ hypotheses when H_0 is true. This error probability is called Type-1 error probability. If the likelihood ratio test is written, we see that the probability of error can be reduced to,

$$P_e = P\left(\left|\boldsymbol{r}^H \boldsymbol{a}_{\psi_0}\right|^2 < \left|\boldsymbol{r}^H \boldsymbol{a}_{\psi_0 + h}\right|^2\right) \tag{21}$$

$$P_e = P(|B(\psi_0)|^2 < |B(\psi_0 + h)|^2) \tag{22}$$

$$\begin{bmatrix} B(\psi_0) \\ B(\psi_0 + h) \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_{\psi_0}^H \\ \boldsymbol{a}_{\psi_0+h}^H \end{bmatrix} \boldsymbol{r} = \mathbf{A}^H \boldsymbol{r} \tag{23}$$

$$\boldsymbol{r} \sim CN\left(\mu_\eta \boldsymbol{a}_{\psi_0}, 2\sigma_\eta^2 \boldsymbol{a}_{\psi_0} \boldsymbol{a}_{\psi_0}^H + \sigma_n^2 \boldsymbol{I}\right) \tag{24}$$

$$\boldsymbol{r} \sim CN(\boldsymbol{\mu_r}, \boldsymbol{C_r}) \tag{25}$$

$$\begin{bmatrix} B(\psi_0) \\ B(\psi_0 + h) \end{bmatrix} \sim CN(\boldsymbol{A}^H \boldsymbol{\mu_r}, \boldsymbol{A}^H \boldsymbol{C_r} \boldsymbol{A}) \tag{26}$$

We need the probability of the event that compares the magnitude squares of two jointly Gaussian distributed random variables. This analysis is given in literature [9], [10] and can be summarized, using the Stein's notation, as follows,

$$\bar{z}_{if} = m_{if} + j\mu_{if} = \left|\bar{z}_{if}\right| e^{j\theta_{if}},\ \ i = 1,2 \tag{27}$$

$$S_{if} = \tfrac{1}{2}\left|\bar{z}_{if}\right|^2 = \tfrac{1}{2}\left(m_{if}^2 + \mu_{if}^2\right) \tag{28}$$

$$N_{if} = \tfrac{1}{2}\left|z_{if} - \bar{z}_{if}\right|^2 \tag{29}$$

$$\rho_f = \frac{1}{2\sqrt{N_{1f}N_{2f}}}\overline{\left(z_{1f} - \bar{z}_{1f}\right)^*\left(z_{2f} - \bar{z}_{2f}\right)} \tag{30}$$

$$\phi = \arg\left(\rho_{cf} + j\rho_{sf}\right) \tag{31}$$

$$\left\{\begin{matrix} a \\ b \end{matrix}\right\} = \frac{1}{2}\left[\frac{S_{1f}+S_{2f}+2\sqrt{S_{1f}S_{2f}}\cos(\theta_{1f}-\theta_{2f}+\phi)}{N_{1f}+N_{2f}+2\sqrt{N_{1f}N_{2f}}\left|\rho_f\right|^2} + \frac{S_{1f}+S_{2f}-2\sqrt{S_{1f}S_{2f}}\cos(\theta_{1f}-\theta_{2f}+\phi)}{N_{1f}+N_{2f}-2\sqrt{N_{1f}N_{2f}}\left|\rho_f\right|^2} \mp \frac{2(S_{1f}-S_{2f})}{\sqrt{\left(N_{1f}+N_{2f}\right)^2 - 4N_{1f}N_{2f}\left|\rho_f\right|^2}}\right] \tag{32}$$

$$A = \frac{N_{1f}-N_{2f}}{\sqrt{\left(N_{1f}+N_{2f}\right)^2 - 4N_{1f}N_{2f}\left|\rho_f\right|^2}} \tag{33}$$

$$P\left(\left|z_{1f}\right| < \left|z_{2f}\right|\right) = \tfrac{1}{2}\left[1 - Q_1\left(\sqrt{b},\sqrt{a}\right) + Q_1\left(\sqrt{b},\sqrt{a}\right)\right] - \tfrac{A}{2}\exp\left(-\tfrac{a+b}{2}\right) I_0\left(\sqrt{ab}\right) \tag{34}$$

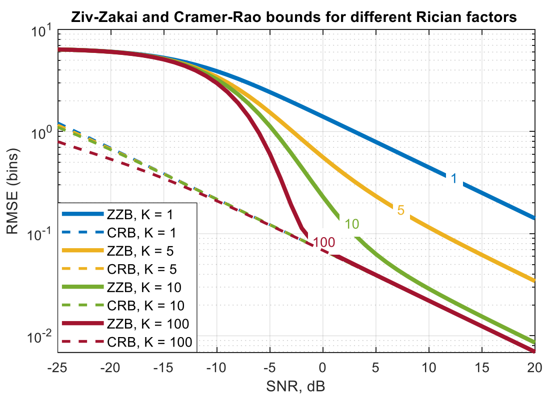

Fig. 3. Comparison of Cramer-Rao and approximate deterministic Ziv-Zakai Bounds under different fading conditions.

Consequently, for the bearing angle estimation accuracy problem, we can use the Stein's probability of error formulation to calculate the approximate deterministic Ziv-Zakai Bound.

The practical advantage of using Stein's analysis for error analysis is that we have a closed form expression for the probability of error for different fading conditions (for any Rician factor). In Fig. 3, Cramer-Rao and deterministic Ziv-Zakai bounds are calculated for different Rician factors and SNR values, for a fine frequency estimation problem. As seen in this figure, the changes in the Cramer-Rao bounds in response to the drastic changes in the fading conditions is almost negligible. However deterministic Ziv-Zakai bound is a tighter bound, as emphasized by Monte Carlo results in Fig. 4 for fine frequency estimation problem (which is analogous to direction finding problem using a uniform line array with an element spacing of λ/2). It is quite straightforward to apply the ZZB to the direction finding performance estimation.

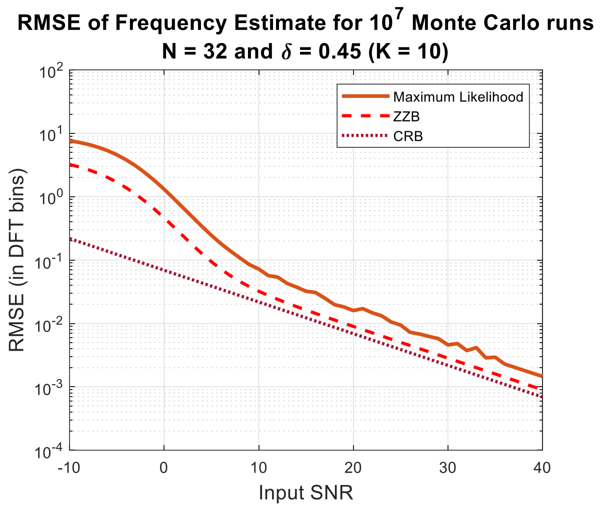

Fig. 4. Comparison of Cramer-Rao and deterministic Ziv-Zakai Bounds with 10^7 Monte Carlo simulations using a Rician factor (K) of 10.

As a case study for the direction of arrival problem, we examine a 12 element circular array with directional elements as depicted in Fig. 5. Each sensor is modeled to have a cosine pattern with the following expression;

$$B_n(\phi) = \left|\tfrac{1}{2}\cos(\phi - \phi_n) + \tfrac{1}{2}\right|^2 \tag{35}$$

$$\phi_n = \frac{(n-1)\pi}{6}\ ,\ n = 1,2,\ldots,12. \tag{36}$$

Sensors are placed with equal angular separations on a circle with radius of 0.5 meters for an operating frequency of 10 kHz and 1500 m/sec sound speed. With these definitions, the approximate deterministic Ziv-Zakai bound is calculated for different input SNR and target DOA values for two different Rician factor values. The results are presented in Fig. 6 and Fig. 7, for K → ∞ (no fading) and K = 10 respectively. Expectedly, we observe a very close performance for all directions, due to the symmetric array structure. Notice that the threshold SNR is about 1.5 dB worse when K = 10 compared to K → ∞.

Although it is not apparent in the figures, the asymptotic performance is slightly worse for K = 10 as well.

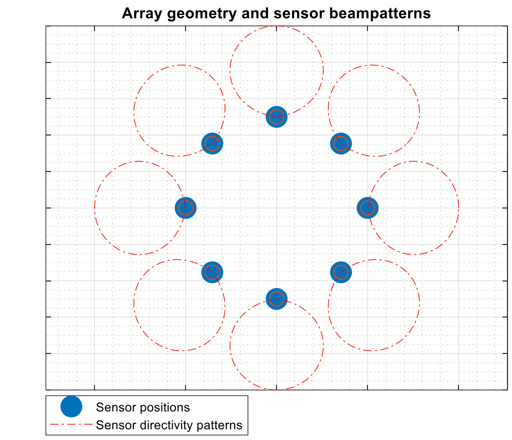

Fig. 5. Array geometry and sensor beampatterns when all sensors are enabled.

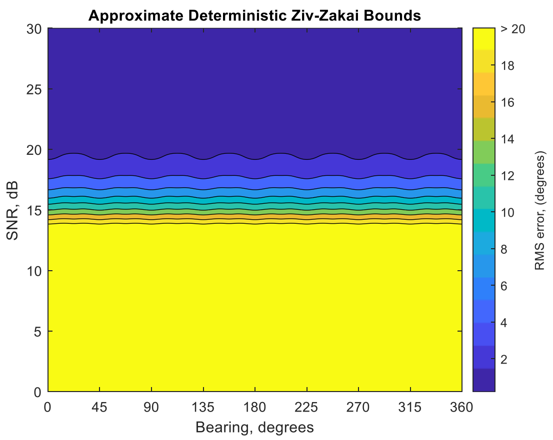

Fig. 6. Approximate Deterministic Ziv-Zakai Bounds for different input SNR and bearing values when all sensors are enabled and no fading ($K \to \infty$).

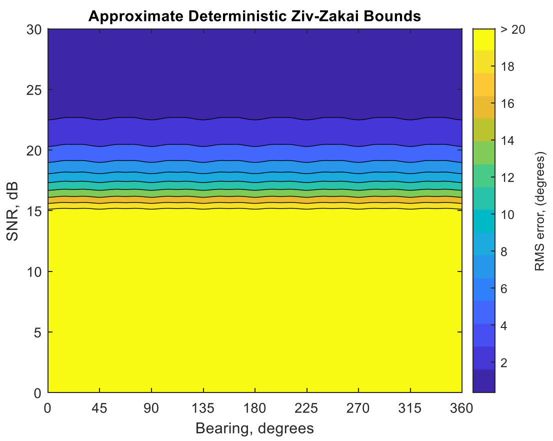

Fig. 7. Approximate Deterministic Ziv-Zakai Bounds for different input SNR and bearing values when all sensors are enabled for $K = 10$.

When we disable two sensors (Fig. 8), the performance expectedly deteriorates within the angular sector of the disabled sensors as seen in Fig. 9. Again we observe an increase in the threshold SNR about 1.5 dB when K = 10 compared to K → ∞.

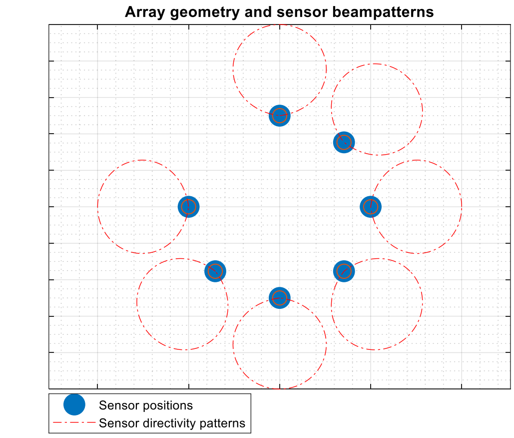

Fig. 8. Array geometry and sensor beampatterns when two sensors are disabled.

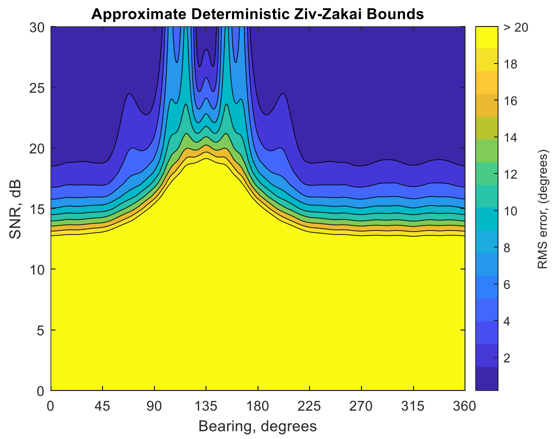

Fig. 9. Approximate Deterministic Ziv-Zakai Bounds for different input SNR and bearing values when a sensor is disabled and no fading ($K \to \infty$).

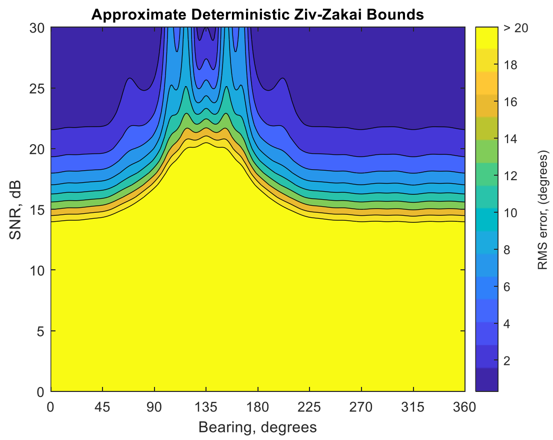

Fig. 10. Approximate Deterministic Ziv-Zakai Bounds for different input SNR and bearing values a sensor is disabled for $K = 10$.

4 Conclusion

In this paper we have given the closed form expressions for the calculation of approximate deterministic Ziv-Zakai lower bound under Rician fading conditions. These bounds are applied in the performance prediction of arrays with disabled sensors due to faults. Utilization of approximate deterministic Ziv-Zakai bounds simplifies the performance analysis, due to the availability of closed form expressions for the probability calculation of an error event required for the bound. Consequently, for a given array geometry and fading parameters, it is possible to present to the sonar operator an online performance prediction tool. This provides the sonar operator an ability to almost instantaneously assess the array performance with disabled sensors. The analysis in this paper is carried out specifically for the direction finding applications, however the derivations are analogous to the frequency estimation problem and can be trivially extended to this problem as well.

References

[1] A H. L. V. Trees, Detection, “Estimation and Modulation Theory”, John Wiley and Sons Inc., 1968.
[2] H. L. V. Trees, K. L. Bell, “Bayesian bounds for parameter estimation and nonlinear filtering tracking”, Wiley-IEEE Press, 2007.
[3] E. W. Barankin, “Locally best unbiased estimates”, Ann. Math. Statist., vol. 20, pp. 477-501, 1949.
[4] A. J. Weiss, E. Weinstein, “Fundamental limitations in passive time delay estimation – Part I: Narrow-band systems”, IEEE Trans. Acoust. Speech Signal Process., vol. 31, no. 2, pp. 472-486, April 1983.
[5] J. Ziv, M. Zakai, “Some lower bounds on signal parameter estimation,” IEEE Trans. Inf. Theory, vol. 15, no. 3, pp. 386–391, May 1969.
[6] K. K. Bell, Y. Eprahim, and H. L. V. Trees, “Explicit Ziv-Zakai lower bound for bearing estimation,” IEEE Transactions on Signal Processing, vol. 44, no. 11, 1996.
[7] S. M. Kay, Fundamentals of Statistical Signal Processing, Volume 1: Estimation Theory. Prentice-Hall, 1993.
[8] D. Khan and K. L. Bell, “Explicit Ziv-Zakai bound for analysis of DOA estimation performance of sparse linear arrays,” Elsevier Signal Processing, vol. 93, pp. 3449–3458, 2013.
[9] S. Stein, “Unified analysis of certain coherent and noncoherent binary communication systems,” IEEE Transactions on Information Theory, vol. IT-10, pp. 43–51, 1964.
[10] M. K. Simon, M. S. Alouni, Digital Communication over fading channels: A unified approach to performance analysis. John Wiley and Sons, 2000.

Author/Speaker Biographies

Erdal MEHMETCIK received his B.S. and M.S. degrees from Electrical-Electronics Engineering Department of Middle East Technical University, in 2009 and 2011 respectively. He is currently a Ph.D. Candidate in the same department and works as a sonar systems algorithm engineer in ASELSAN Inc. since 2011. His research interests include sonar signal processing.

Çağatay CANDAN is a professor at Electrical-Electronics Engineering Department of Middle East Technical University. He received his B.S., M.S., and Ph.D. degrees, from Middle East Technical University (1996), Bilkent University (1998) and Georgia Institute of Technology (2004), respectively. His research interests include statistical signal processing and its applications in array processing.